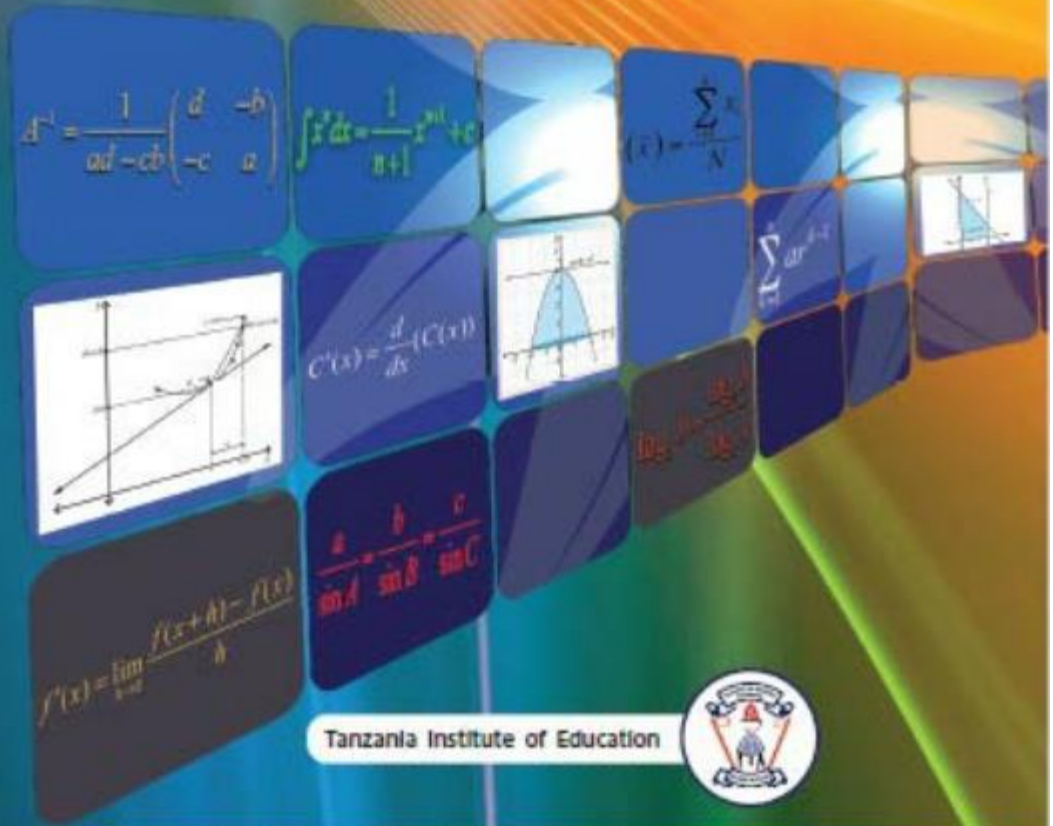
Basic Applied Mathematics

for Advanced Level Secondary Schools

Student's Book

Form Five and Six



Tanzania Institute of Education

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Basic Applied Mathematics

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Form Five and Six

THE UNITED REPUBLIC OF TANZANIA
MINISTRY OF EDUCATION,
SCIENCE AND TECHNOLOGY

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Preface

This book, *Basic Applied Mathematics for Advanced Level Secondary Schools* **is** written specifically for Form Five and Six students in the United Republic of Tanzania. **The book is prepared according to the 2009 Basic Applied Mathematics Syllabus for Advanced Level Secondary Education Form V-VI, issued by the Ministry of Education and Vocational Training.**

The book consists of eleven chapters, which are: Calculating devices, Functions, Algebra, Differentiation, Integration, Statistics, Probability, Trigonometry, Exponential and logarithmic functions, Matrices, and Linear programming. In addition to the content, each chapter contains activities, illustrations, exercises, and revision questions. Learners are encouraged to do all activities and answer all questions so as to enhance their understanding, and promote acquisition of the intended skills, knowledge, and attitudes.

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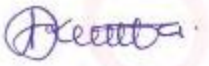

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Chapter
One

Calculating devices

Introduction

The knowledge of calculating devices has evolved through a number of important stages of its development; from the stage of counting with fingers and toes to the use of electronic calculators and computers. Even with the advent of today's powerful electronic calculating machines, some canonical devices such as the slide rule and Abacus are still in use. In this chapter, you will learn how to use calculators and computers. In using calculators, you will learn the basic features of non-programmable scientific calculators, and the steps for operating some of its basic keys in different mathematical operations. With computers, you will learn how to use the computer calculator and the spreadsheet.

1.1 Scientific calculators

A scientific calculator is an electronic device used in calculating or computing mathematical expressions and equations. It is usually designed to handle computations in disciplines such as Science, Business and Economics, Humanities and Mathematics. Its functions have almost replaced slide rules and mathematical tables which were basic devices in traditional calculations and widely used by students and other professionals such as bankers, and others.

There are two types of scientific calculators, namely programmable and non-programmable. They differ from each other in terms of their functionalities. Programmable scientific calculators have the ability to graph input data, write, and store programs, while non-programmable scientific calculators are not capable of graphing input data, write, or store programs.

Most of these scientific calculators have two-line display which enable the user to see both the input under operation and the output (result). Scientific calculators are of different models, manufactured from various companies such as Hewlett-Packard, Texas instruments, Casio, Sharp, and Canon. Non-programmable scientific calculators are recommended for use in this book.

Basic features of non-programmable scientific calculators

Scientific calculators have many features that may differ from one manufacturer to another as well as from one model to another. However, their basic operations remain the same. Figure 1.1 shows the basic features of the non-programmable scientific calculator while Table 1.1 displays some basic symbols/ keys and their functions.

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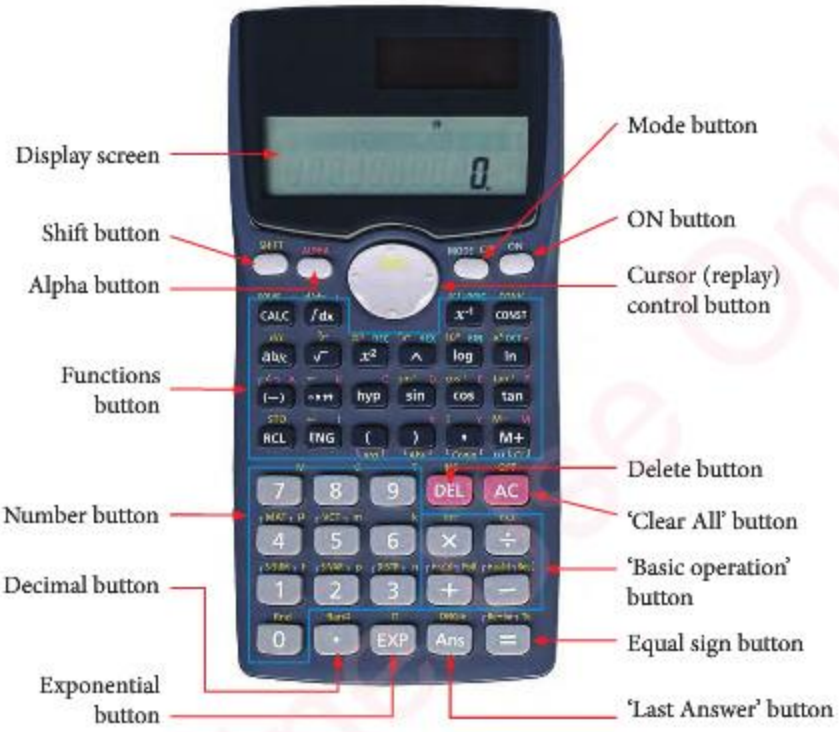

Figure 1.1 Basic features of a non-programmable scientific calculator

Table 1.1 Keys and functions of a non-programmable scientific calculator

Symbols/Keys	Name	Function	Example
ON/OFF keys	Switching on/off keys	Used for making a calculator ready (on) for use or shut down (off) after being used	
+, −, ×, ÷	Basic arithmetic operations	Addition, subtraction, multiplication, and division of numbers	$2+4-5\times6\div3$
x^2, x^3, ^	Exponents	Used for calculating the square, cubic, and numbers raised to other exponents, respectively	$5^2, 6^3, 4\text{^}8$
=	Equal sign	Used for displaying an answer of a certain operation	$2+4=6$

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$\sqrt{\ }, \sqrt[3]{\ }, \sqrt[x]{\ }$	Roots	Used for calculating square root, cube root, and x^{th} roots of numbers, respectively	$\sqrt{4}$, $\sqrt[3]{28}$, $\sqrt[3]{120}$
log, ln	Logarithmic and Natural logarithmic function keys	Used for calculating logarithm and natural (Naperian) logarithm of numbers respectively	log 5, ln 4
$e^x, 10^x$	Exponential function keys	Used for finding the inverse of natural logarithm of a number and antilogarithm to base 10, respectively	$e^{0.6931}$, $10^{0.06786}$
sin, $\sin^{-1}$, cos, $\cos^{-1}$, tan, $\tan^{-1}$	Trigonometric function keys	Used for finding sine, cosine, and tangent values of an angle and their respective inverses	$\sin 40°$, $\sin^{-1}(0.5)$, $\cos 30°$, $\cos^{-1}(0.86)$, $\tan 50°$, $\tan^{-1}(1.19)$

Activity 1.1

Apart from the functional keys explained in Table 1.1:

1. Identify other four useful functional keys on a non-programmable scientific calculator.
2. Show how the keys mentioned in (1) can be used to calculate the values of the functions under consideration.

Steps for operating some basic keys of non-programmable scientific calculators

In this chapter, a CASIO *fx - 991 MS* will be used to perform the calculations. This choice was made due to the fact that CASIO calculators are easily found in the market. But other types of calculators may be used to perform the same computation even though their operations may differ from that of CASIO *fx - 991 MS*.

a) Replay (Cursor control) button

Scientific calculators such as CASIO *fx - 991 MS* have a replay button which is used to review calculations. The replay button is marked with four arrows used for shifting the cursor to the left, right, top, and bottom by considering the direction of the arrows. The "REPLAY" button for some non-programmable calculators have a "COPY" as another function.

b) Preceding keys

Scientific calculators have preceding keys which are used to precede the other keys depending on their colours. There are two preceding keys in scientific calculators, namely "SHIFT" key and "ALPHA" key. "SHIFT" key precedes all keys with yellow colour while "ALPHA" key precedes all keys with pink colour to access their functions. The steps for using these keys are as follows:

Step 1: Press "SHIFT" or "ALPHA" key.

Step 2: Press the key corresponding to the colour which is preceded by "SHIFT" or "ALPHA" keys.

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Example 1.1: Calculate $\sqrt[5]{120}$ by using a non-programmable scientific calculator.

Solution

Step 1: Type 5 using the function button of the calculator.

Step 2: Press "SHIFT" key.

Step 3: Press " $\sqrt[x]{\ }$ " key.

Step 4: Type 120 using the number buttons of the calculator.

Step 5: Press "=" button to obtain the result.

On completing the above steps, the result for $\sqrt[5]{120}$ is 2.605171085.

c) *Memory features and brackets*

Scientific calculators can store answers for a while and retrieve (recall) when needed for use in solving a certain mathematical problem.

The following steps are used in storing result for a while in a non-programmable scientific calculator.

Step 1: Type a number to be stored.

Step 2: Press "SHIFT" key.

Step 3: Press "STO" key and then "M+" key to store the number displayed.

Step 4: Press "RCL" key followed by "M+" key to recall the stored number.

Step 5: Press " = " key to obtain the required result.

Example 1.2: Use the memory features of a scientific calculator to evaluate

$$\frac{100}{-21.5+1.5}$$

by using a non-programmable scientific calculator.

Solution

By using memory features;

Step 1: Type –21.5 and press "+" key.
Step 2: Type 1.5 and press " = " key.

Step 3: Press "SHIFT" key.

Step 4: Press "STO" key and then press "M+" key.
Step 5: Type 100 and press "÷" key.

Step 6: Press "RCL" key and then press "M+" key.
Step 7: Finally, press " = " key to obtain the result.

After following the above steps correctly, the result is –5.

Alternative approach:

Step 1: Type 100 and then press "÷" key.

Step 2: Press opening bracket "(" key, type –21.5, press addition "+" key, type 1.5, and then press closing bracket ")" key.

Step 3: Finally, press "=" button to get the result.

After following the above steps correctly, the result is –5.

d) *Scientific notation*

Scientific notation, also referred to as standard form or standard index form is a way of expressing numbers in the form $m\times10^n$, where the exponent n is an integer and $1 \le m < 10$. By using scientific calculators such as CASIO *fx - 991 MS*, numbers can be

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expressed in scientific notation. The steps for this operation are as follows:

Step 1: Type the number using number buttons.

Step 2: Press the equal "=" button.

Step 3: Press the "ENG" button.

Example 1.3: Express 1234 in scientific notation by using a non-programmable scientific calculator.

Solution

Step 1: Type 1234 using the number buttons of the calculator.

Step 2: Press the "=" button.

Step 3: Press the "ENG" button.

After following the above steps, the result is 1.234×10^{03}.

Remark: To express a number in scientific notation with equal or more number of digits than the number limit of the calculator, type the number then press the "=" button to get the number in the exponential form.

For example, for CASIO *fx - 991 MS,* the maximum digits which can be displayed on the calculator's display screen without scrolling is 11. Thus, to express a numeral like 13472894723 (numeral with 11 digits) or 1347289472359 (numeral with 13 digits) in scientific notation, just type the given numerals and then press the equal button "=". In this case, 13472894723 will be displayed as $1.3472894723\times10^{10}$ and 1347289472359 will be expressed as $1.347289472359\times10^{12}$.

Also, to express the number in scientific notation with a specified number of significant figures, the following steps are used:

Step 1: Press the "MODE" key five times.

Step 2: Select "Sci" mode by pressing the key numbered 2 as instructed on the calculator's screen

Step 3: Type the number of significant figures from the given limits 0 ~ 9, where 0 stands for 10 significant figures.

Step 4: Type the number using number buttons or perform any mathematical calculations.

Step 5: Press "=" button to get the result in scientific notation.

Example 1.4: Use a non-programmable scientific calculator to express the following in scientific notation with 3 significant figures:

a) 0.0017298 b) $\dfrac{675\times38}{100001}$

Solution

Using the steps above.

a) $0.0017298=1.73\times10^{-3}$

b) $\dfrac{675\times38}{100001}=2.56\times10^{-1}$

e) Logarithmic functions for base 10 and base e

Scientific calculators are able to find common logarithm (base 10) and natural logarithm (base e) of numbers and their respective inverses.

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The following steps are used to find the logarithm of a number to base 10:

Step 1: Press the "log" key.

Step 2: Write the number whose logarithm is to be determined.

Step 3: Press the "=" key. Then the logarithm of the number will be displayed.

1.5: Evaluate log 10000 by using a non-programmable scientific calculator.

: Step 1: Press the "log" key

Step 2: Type 10000

Step 3: Press the "=" key

After following the above steps, the result is 4.

10

Steps on how to use a calculator to find the anti-logarithm of a number to base 10 are:

Step 1: Press "SHIFT"key.

Step 2: Press the "log" key.

Step 3: Write the number whose antilogarithm is required.

Step 4: Press the "=" button key.

1.6: Find the anti-logarithm of 0.0124 by using a non-programmable scientific calculator.

: Step 1: Press "SHIFT"key.

Step 2: Press the "log" key.

Step 3: Write 0.0124.

Step 4: Press the "=" button key.

After following the above steps, the result is 1.028963572

(e)

The natural logarithm of a number is its logarithm to the base of the mathematical constant e, where "e" is an irrational number approximately equal to 2.718281828459. The natural logarithm of x is generally written as $\ln x$ or $\log_e x$.

Steps on how to use a calculator to find the natural logarithm of a number are:

Step 1: Press the "ln" key.

Step 2: Write the number whose natural logarithm is required.

Step 3: Press the "=" key to obtain the result.

After following these steps, natural logarithm of 247 is written as ln 247 and its value is displayed as 5.509388337 .

: Anti-logarithms of natural logarithms are obtained by using the same steps as those of base 10 but in this case, the "ln" key is used.

f) Degree button

The degree button "° ' '' " is used to insert the numbers of degree format into the calculator. A number in degree format is divided into three parts, which are Degree "°", Minutes "' ", and Seconds "'' " .

Steps for inserting numbers into degree format are:

Step 1: Enter the degree part of the number followed by " ° ' '' " button.

Step 2: Enter the minutes part of the number and then "° ' '' " button.

Step 3: Enter the seconds part of the number followed by the "° ' '' " button.

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Step 4: Press the "=" button to obtain the result in degree format.

Example 1.7: Use a non-programmable scientific calculator to write the following numbers in their respective form:

a) 15°40'59" b) 5'49"

c) 68" d) 30°79"

Solution: After following the steps for inserting numbers into degree format;

a) 15°40'59" = 15°40'59"

b) 5'49" = 0°5'49"

c) 68" = 1°8'0"

d) 30°79" = 31°19'0"

A number in degree format can be converted to radian format and vice versa by using a non-programmable scientific calculator. The conversion involves the relationship between radian and degree. That is;

2π (radians) = 360° which simplifies to; π (radians) = 180°.

Steps to convert a number in degree format to radian.

Step 1: Change the calculator mode into radians by pressing the "MODE" button four times and then press "2" button key.

Step 2: Write the number you want to convert to radians.

Step 3: Press "SHIFT" key followed by "ANS" key.

Step 4: Press "1" button key to place a degree symbol on a number on the screen.

Step 5: Press "=" button key to obtain the result in radian format.

Remark: In order to resume to the normal mode (degree format), press the "MODE" button four times and then press "1" button to choose degree.

Steps to convert numbers in radian to degree format.

Step 1: Make sure your calculator is in radian degree mode by pressing the "MODE" button four times and then press "1" button.

Step 2: Enter the radian number.

Step 3: Press "SHIFT" key followed by "ANS" key.

Step 4: From the menu, press "2" button followed by "=" to display the answer in degrees.

Example 1.8: Use a non-programmable scientific calculator to convert each of the following numbers to their respective form:

a) 30°15'40" to radian format

b) 0.8809 to degree format

Solution: Using the listed steps

a) 30°15'40" = 0.528156024

b) 0.8809 = 50°28'18.67"

g) Trigonometric functions

Values of trigonometric ratios such as sine, cosine, tangent, and their inverses can be determined using scientific calculators. For CASIO *fx - 991 MS*, the steps for computing values of trigonometric ratios are as follows:

Step 1: With the scientific calculator in degree mode "D Mode", press the key of the trigonometric ratio required; for example, sin, cos, tan .

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Step 2: Write the angle whose trigonometric ratio is required.

Step 3: Press the "=" key.

Remark: If the trigonometric function has an angle in radian format for example, 2 radians, $\frac{2\pi}{3}$ radians or if the function involve the inverse of trigonometric function, then the mode of calculator must be "Rad".

Example 1.9: Use a non-programmable scientific calculator to evaluate each of the following trigonometric ratios:

a) $\sin 13°$ b) $\cos 57°$

c) $\tan 94°$ d) $\sin 3$ where the angle is in radian

e) $\cos\frac{2}{3}\pi$ f) $\cos^{-1}(0.134)$

Solution

a) $\sin 13° = 0.224951054$

b) $\cos 57° = 0.544639035$

c) $\tan 94° = -14.30066626$

d) $\sin 3 = 0.141120008$

e) $\cos\frac{2}{3}\pi = -0.5$

f) $\cos^{-1}(0.134) = 1.436392034$

Activity 1.2

1. Identify useful keys/buttons on a non-programmable scientific calculator used for determining trigonometric ratios and their inverses.
2. Write the steps for finding the inverse of trigonometric ratios using a non-programmable scientific calculator.
3. Set a non-programmable scientific calculator in the radian (Rad) mode, and then with the aid of an example, outline the steps for finding the trigonometric ratio of an angle expressed in degrees.

h) Exponential functions and roots

Scientific calculators can be used to express numbers in exponential form or in x^{th} root form "$\sqrt[x]{\ }$" and find their corresponding values. The buttons used are "$\wedge$" and "SHIFT". The steps for expressing a number in exponential form are as follows:

Step 1: Enter the number.

Step 2: Press the "$\wedge$" button followed by the exponent.

Step 3: Press the "=" button to obtain the value of the exponential number.

The steps for finding the x^{th} root of a number using a scientific calculator are as follows:

Step 1: Enter the root index number (e.g. 2 for square root, 3 for cube root, 4 for fourth root etc).

Step 2: Press the "SHIFT" button.

Step 3: Press the "$\wedge$" button and the symbol $\sqrt[x]{\ }$ will be displayed.

Step 4: Enter the number whose root is required.

Step 5: Press the "=" button to get the required result.

Example 1.10 Use a non-programmable scientific calculator to evaluate each of the following numbers:

a) $\sqrt[5]{243}$ b) $\sqrt[11]{9500}$ c) $(5.3)^2$

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Solution

a) Following the listed steps, the displayed value of $\sqrt[5]{243}$ is 3

b) Following the listed steps, the displayed value of $\sqrt[11]{9500}$ is 2.29938258

c) Following the listed steps, the displayed value of $(5.3)^2$ is 28.09

i) Operations on Fractions

Scientific calculators can perform operations on fraction using a fraction button key. For example CASIO *fx - 991 MS*, the fraction basic button key is labelled as "$a\ b/c$".

The following steps can be used to write a fraction in the scientific calculator:

Step 1: Type the numerator.

Step 2: Press the fraction button key, that is "$a\ b/c$".

Step 3: Type the denominator.

Step 4: Press the "=" button key to obtain the result in fraction format.

The following steps can be used to write a mixed fraction in the scientific calculator:

Step 1: Type the whole number part.

Step 2: Press the fraction button key, that is "$a\ b/c$".

Step 3: Type the numerator.

Step 4: Press the fraction button key, that is "$a\ b/c$".

Step 5: Type the denominator.

Step 6: Press the "=" button key to obtain the result in fraction format.

Example 1.11: Use a non-programmable scientific calculator to obtain the value of the following:

a) $1\frac{2}{3}+3\frac{4}{5}-2\frac{1}{4}$

b) $\frac{13}{5}\ \ \frac{2}{7}$

Solution After following the above steps,

a) $1\frac{2}{3}+3\frac{4}{5}-2\frac{1}{4}=3\frac{13}{60}$

b) $\frac{13}{5}\div\frac{2}{7}=9\frac{1}{10}$

Scientific calculators can also perform operations of converting numbers from fractions to decimals and vice versa.

The following steps can be used to convert fractions to decimal numbers:

Step 1: Type the numerator.

Step 2: Press the fraction button key, that is "$a\ b/c$".

Step 3: Type the denominator.

Step 4: Press the "=" button key to obtain the result in fraction format.

Step 5: Press the fraction button again, that is "$a\ b/c$" button key to obtain the result in decimal format.

When converting numbers from decimal to fraction, apply the following steps:

Step 1: Type the decimal number.

Step 2: Press the "=" button key to obtain the result in decimal form.

Step 3: Press the fraction button key, that is "$a\ b/c$" to obtain the result in fraction form.

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Example 1.12: Use a non-programmable scientific calculator to convert each of the following numbers to their respective forms.

a) $2\frac{3}{5}$ decimal form.

b) 5.23 into fraction form.

Solution: Follow the listed steps to get

a) $2\frac{3}{5}$ is equal to 2.6 in decimal form.

b) 5.23 is equal to $5\frac{23}{100}$ in fraction form.

j) Statistical calculations

In a non-programmable scientific calculator, numerical statistical data can be processed to determine various statistical values such as sum, mean, and standard deviation for ungrouped and grouped data.

Ungrouped statistical data

Apply the following steps for ungrouped statistical data:

Step 1: Press the "MODE" key twice

Step 2: Select "SD" mode by pressing the button numbered 1 as instructed on the screen of the calculator.

Step 3: Type the number (value) then press "M+" button key.

Step 4: Repeat step (3) for each of the values.

After inserting the data into the calculator on the "SD" mode, the steps to determine sum of data values $\sum x$, sum of squared values $\sum x^2$, mean $\bar{x}$, and standard deviation δ_x are as follows:

Step 1: Press "SHIFT" button.

Step 2: Press the button key numbered 1 or 2 to obtain different options of statistical values to be obtained.

Step 3: Select a parameter needed by pressing the relevant button as instructed on the screen of the calculator.

Step 4: Press the "=" button to obtain the result.

Example 1.13: Given the data 5, 4, 7, 3, 2, and 9. Use a non-programmable scientific calculator to find the following:

a) Mean

b) Standard deviation

c) Sum of data values

Solution

Following the required steps, the results are as follows:

a) Mean, $\bar{x} = 5$

b) Standard deviation

$\delta_x = 2.380476143$

c) Sum of data values, $\sum x = 30$

Grouped statistical data

Apply the following steps for grouped statistical data:

Step 1: Press the "MODE" key twice.

Step 2: Select "SD" Mode by pressing the key numbered 1 as instructed on the calculator's screen.

Step 3: Type the class mark data.

Step 4: Press "SHIFT" button followed by the comma to obtain " ; ".

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Step 6: Type the corresponding frequency of the class mark.

Step 7: Press "M+" button.

Step 8: Repeat steps 3 – 7 for each class mark and its corresponding frequency.

Step 9: Press "SHIFT" button, then press button key numbered 1 or 2 to obtain different options of statistical value.

Step 10: Select a statistical value needed by pressing the relevant button as instructed on the screen of the calculator.

Step 11: Press the "=" button to obtain the result.

Example 1.14: Table 1.2 shows the frequency distribution for grouped data. Find the mean and standard deviation by using a non-programmable scientific calculator.

Table 1.2 Frequency Distribution Table

Class interval (C.I)	Class mark (x)	Frequency (f)
11 – 20	15.5	12
21 – 30	25.5	8
31 – 40	35.5	9
41 – 50	45.5	10
51 – 60	55.5	5
61 – 70	65.5	6

Solution: Following the above steps, the results are:

a) Mean, $\bar{x} = 36.7$

b) Standard deviation, $\delta_x = 16.69011684$

Remarks

1. Once statistical data are stored in the scientific calculator, they cannot be cleared by turning off the calculator or by pressing the ON or AC keys. The data can only be erased by either changing the MODE or by running the steps for clearing statistical data.
2. To clear the data from the scientific calculator, the steps are:
   Step 1: Press "SHIFT" key.
   Step 2: Press "MODE" key.
   Step 3: Press "3".
   Step 4: Press the "=" button key twice.

Activity 1.3

1. Insert the data 7, 8, 6, 13, 6, 12, 8, 6, 11, 7, 9 in a non-programmable scientific calculator, and then identify the steps and use them to compute the corresponding value for each of the following:
   a) Sum of data values
   b) Sum of squares of data values
   c) The mean
2. Use a non-programmable scientific calculator to find the mean and standard deviation of the following test scores of 68 Form Five students.

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Table 1.3 Grouped test scores

Marks	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50
Frequency (f)	4	13	21	20	10

3. Outline the steps for:
   a) Editing statistical data entered in a non-programmable scientific calculator.
   b) Clearing statistical data in a non-programmable scientific calculator.

k) Probability Calculations

The scientific calculator can also be used to perform various probability calculations.

Combination of items

To find the number of n combinations in groups of r items by using a non-programmable scientific calculator, CASIO *fx - 991 MS* use the following steps:

Step 1: Type the n value.

Step 2: Press "SHIFT" button.

Step 3: Press the "÷" key to obtain ^nC_r function.

Step 4: Type the r value to complete the expression for combination (i.e. ^nC_r).

Step 5: Press the "=" key to obtain the result.

Permutations of items

To find the number of permutations of n items in groups of r items using a non-programmable scientific calculator such as CASIO *fx - 991 MS* use the following steps:

Step 1: Type the n value.

Step 2: Press "SHIFT" button.

Step 3: Press the "×" button to obtain ^nP_r function.

Step 4: Type the r value to complete the expression for permutation (i.e ^nP_r).

Step 5: Press the "=" button to obtain the result.

Example 1.15: By using a non-programmable scientific calculator, write the formulae of the given permutation and combination and then write their corresponding values that will be displayed on the calculator's screen.

a) Permutation of 11 items in a group of 6

b) Combination of 7 items in a group of 3

c) Combination of 8 items in groups of 3 items

d) Permutation of 5 items in groups of 4 items.

Solution

The corresponding values displayed are:

a) $^{11}P_6$, the corresponding value is 332640.

b) $^7C_3$, the corresponding value is 35.

c) $^8C_3$, the corresponding value is 56.

d) $^5P_4$, the corresponding value is 120.

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Activity 1.4

1. Outline the steps for evaluating the factorial of a number by using a non-programmable scientific calculator.
2. Use a non-programmable scientific calculator to evaluate the following:

   a) 5! b) 7! – 2! c) $\frac{3!}{4!2!}$

l) Solutions to Simultaneous, quadratic, and cubic equations

In non-programmable scientific calculators, systems of simultaneous equations with two or three unknowns, quadratic and cubic equations can be solved by following some operation steps.

The steps for solving simultaneous equations are:

Step 1: Press the "MODE" button three times until the option for choosing "EQN" (equation) appears on the screen.

Step 2: Choose "EQN" by pressing the key of its option number.

Step 3: Choose the number of unknowns, 2 for simultaneous equations of two unknowns or 3 for simultaneous equations of three unknowns which will appear on the screen. Press the number key corresponding to the number of unknowns in the system of simultaneous equations. That is either the key labelled 2 or 3. Coefficient of the unknown x, y and z in the equations are named as follows:

The system of linear simultaneous equations in two unknowns has the form;

$$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$$

The system of linear simultaneous equations with three unknowns are:

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases}$$

Step 4: Once the number of unknown is set, the calculator's screen will indicate the value of a_1.

Step 5: Insert the number representing the coefficient a_1, and then press the "=" button key.

Step 6: Repeat step 5 for the rest of the coefficients until you obtain their corresponding last constants c_2 for 2-unknowns and d_3 for 3-unknowns.

Remark: After inserting the last constant (c_2 or d_3), the value of the first unknown x will be displayed.

Step 7: Press the "=" button to see the solution of the other unknowns.

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Activity 1.5

Use a non-programmable scientific calculator to solve each of the following system of linear equations

a) $\begin{cases} 5x+3y-4z=9 \\ 2x+3y-z=15 \\ 3x-2y+2z=4 \end{cases}$

b) $\begin{cases} x-2y=4 \\ -3x+5y=-7 \end{cases}$

Steps for solving quadratic and cubic equations

Step 1: Press the "MODE" button three times until the option for choosing "EQN" appears on the screen.

Step 2: Choose "EQN" and press the right arrow of the "REPLAY" button to choose the degree of the equation.

Step 3: Choose the degree of the equation to be solved (i.e. 2 for quadratic equation and 3 for cubic equation).

Step 4: Arrange the equation in an acceptable format for inserting coefficient in the calculator. For quadratic and cubic equations should be in the form $ax^2+bx+c=0$ and $ax^3+bx^2+cx+d=0$, respectively.

Step 5: Insert the coefficient a, and then press the "=" button.

Step 6: Continue with step 5 for coefficients b up to c or d depending on the type of the equation.

Step 7: Once the last value, c or d is inserted in the calculator, the first solution x_1 will be displayed. Press "=" button to get the other solutions.

Activity 1.6

Use a non-programmable scientific calculator to find the solution for each of the following:

1. $2x^2+5x-3=0$
2. $x^3-6x^2-24x+64=0$

m) Calculations involving Matrices

A scientific calculator can store matrices and perform different matrix operations such as addition, subtraction, and multiplication. To store the matrix in the calculator, some steps are required.

Steps for storing matrices in a Scientific Calculator

Step 1: Press the "MODE" key three times until the matrix mode "MAT" appears on the screen and then choose it by pressing its option number.

Step 2: While the calculator is in matrix mode press "SHIFT", then press the 4 key, "MAT".

Step 3: Press 1 and choose the name of the matrix by pressing 1 for naming it matrix $\boldsymbol{A}$, 2 for $\boldsymbol{B}$, and 3 for $\boldsymbol{C}$.

Step 4: Set the dimension "$m\times n$" of the matrix by typing the value of m. Then press "=" button key and type again the dimension n and then press "=" button key. If

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the matrix is 2 by 2, then m is 2 and n is 2 or if the matrix is 3 by 2, then m is 3 and n is 2, etc.

Steps for inserting entries of a Matrix

After the name and dimension of the matrix have been set as described above, entries of the matrix are required. For a scientific calculator such as CASIO *fx - 991 MS*, the following steps may be used to set the entries:

Step 1: Press the first entry "*Mat*A11" if the matrix is named as A. Then, press "=" key.

Step 2: Press the second entry "*Mat*A12" then press "=" key. Follow the same steps until the last entry of the entire matrix.

The entries of matrices are named as illustrated below:

For a 2×2 matrix A, the entries are named as:

$$A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

For a 3×3 matrix A, the entries are named as:

$$A=\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

In other words, A_{11} means an entry of matrix A at the position of row 1 and column 1. Similarly A_{12} refers to the entry of matrix A at row 1 and column 2.

Steps for editing entries of a matrix

Step 1: Press "SHIFT", then press 4, "MAT" button.

Step 2: Select "Edit".

Step 3: Select the name of the matrix whose dimension has been set.

Step 4: Enter the correct value of the entry and then press the "=" button. Continue with these steps until you finish editing all the entries.

Determinant, transpose, and inverse of matrices

After inserting the entries of a matrix, the steps to obtain the determinant of the matrix are as follows:

Step 1: Press "SHIFT" button.

Step 2: Press 4, "MAT" button.

Step 3: Scroll the right arrow of the replay button and select "Det" (Determinant) by pressing number 1 key.

Step 4: Again press "SHIFT" button.

Step 5: Press 4, "MAT" button, then select "MAT" by pressing number 3 key.

Step 6: Select the matrix whose determinant is required.

Step 7: Press the "=" button key to obtain the result (i.e. determinant).

Remark: The same steps 1 – 7 are used for finding transpose of the matrix by choosing transpose "Trn" instead of determinant.

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Inverse of matrices

After getting the result for the determinant of a matrix, use the following steps to determine its inverse:

Step 1: Press "SHIFT" button.

Step 2: Press 4, "MAT" button.

Step 3: Select "MAT" (3) for the matrix.

Step 4: Select the matrix whose inverse is required.

Step 5: "*MatA*" will appear on the screen, then type "x^{-1}" to obtain "*MatA*$^{-1}$" which is the inverse of matrix then press "=" key. The first entry for "*MatAns*11" will appear, continue scrolling the right key until the last answer "*MatAns*33" is displayed.

Activity 1.7

1. Use a non-programmable scientific calculator to do each of the following:

   a) Set matrices $\boldsymbol{A}=\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $\boldsymbol{B}=\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$

   b) Compute the determinant and transpose of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

   c) Compute the inverse of each of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

   d) Write the steps for adding and subtracting the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and give the results.

   e) Describe the steps for calculating the product of matrix $\boldsymbol{A}$ and $\boldsymbol{B}$, and then write the resulting matrix.

2. Use a non-programmable scientific calculator to find $\boldsymbol{C}^{-1}$ if

$$\boldsymbol{C}=\begin{bmatrix} 2 & 6 & 3 \\ 4 & -7 & 9 \\ -5 & 3 & -1 \end{bmatrix}$$

n) Solving definite derivatives and integral of functions

Scientific calculators can be used to calculate the definite derivative and integral of functions. For CASIO *fx - 991 MS*, the basic key for integral of functions is labelled $\int dx$ with its alternate functional key labelled $\frac{d}{dx}$. The functional key $\int dx$ is used for performing calculations which require integration, while $\frac{d}{dx}$ is used for differentiation.

Activity 1.8

Referring to a non-programmable scientific calculator's user guide;

1. Read the manual of a calculator and then write the steps for finding definite derivative and integral of the functions.
2. Using a non-programmable scientific calculator, solve the following;

   a) $\frac{d}{dx}\left(5x^3+4x^2-7x+3\right)$ when $x=2$.

   b) $\int_{-1}^{3}\left(6x^2-3x+5\right)dx$

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o) *Conversion of units*

Unit conversion is one of the functions of a scientific calculator. Non-programmable scientific calculators convert units under metric systems. Types of unit conversion to be carried out by a particular calculator are listed on the user's manual and on the inner part of the plastic cover of the calculator. Thus, before converting the units, identify the number which represent the type of conversion needed.

Steps for unit conversions:

Step 1: Press a number to be converted

Step 2: Press "SHIFT" key

Step 3: Press "CONST"

Step 4: Press a number to recall the unit, example $0.2cm$ into inch

Step 5: Press "=" key

Activity 1.9

1. Identify the types of units' conversion that can be performed by a non-programmable scientific calculator.
2. State the steps for conversion and perform at least three conversions of units of length, mass, time, area, temperature, and pressure.

p) *Physical constants and quick access to constants such as π and e*

Physical constants are useful in calculations. When solving certain problems, sometimes constants are not given hoping that the learners can obviously get them easily from their scientific calculators. Constants which can be recalled by a particular scientific calculator are listed on the inner part of the plastic cover of the calculator as well as in the user's manual. As in the case of units' conversion, each constant is assigned a number for recall. For example, the constant for acceleration due to gravity (g) is assigned number 35. Therefore, to recall it press the button "CONST", then type 35 and press "=" button key to get its value.

Activity 1.10

Use a non-programmable scientific calculator to perform the following;

a) Add any two constants.

b) Subtract any two constants.

c) Multiply any two constants .

d) Divide any two constants.

e) Multiply any three constants.

Exercise 1.1

1. Use a non-programmable scientific calculator to evaluate each of the following. Write the results in standard notation (in three significant figures).

a) $\left[\log_e\left(e^4+2\ln 5\right)+\log 5\right]^5$

b) $\sqrt[6]{\dfrac{(4.03)^3(814765)^{0.5}}{\sqrt{5}}}$

c) $\begin{vmatrix} \left(\dfrac{{}^6C_2\times\ln(0.005)}{\sqrt[3]{43}}\right) & 2e \\ e & \ln 2 \end{vmatrix}$

2. Given $S=\dfrac{1}{y\sqrt{2P}}e^{\left[\frac{-(x-m)^2}{2s^2}\right]}$, where

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$S = 3.08$ and $P = 8$. Use a non-programmable scientific calculator to find the value of y when:

a) $x = m$ b) $x = 1, m = 0$

3. Use a non-programmable scientific calculator to compute the value of the following expression.

$$\log\left|\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{bmatrix}\right| - \log\left|\left(\frac{1+2}{13}\right)^{\frac{1}{3}}\right| \sin(-30°)$$

4. Evaluate the following by using a scientific calculator:

a) $\int_{-2}^{3}(x-2)(x+1)(x-1)\,dx$ b) $\frac{d}{dx}\left(3x^5+7x^2-2e^{-4}\right)$, at $x = 2$

5. Use a non-programmable scientific calculator to perform the following conversions:

a) 100 miles to kilometres
b) 212° F into degrees of Celsius
c) 1000 m into feet

6. Insert the following statistical data in a non-programmable scientific calculator and then find the mean, standard deviation, and the sum of the squares of the data.

Table 1.4 Statistical data

Data	65	66	67	68	69	70
Frequency	2	8	12	17	7	4

7. a) Use a non-programmable scientific calculator to solve the following systems of equations:

i) $\begin{cases} -3x+8y=50 \\ 1\frac{3}{4}x-5y=31.5 \end{cases}$ ii) $\begin{cases} x+2y+4z=2 \\ 2x+6y+6z=3 \\ 3x+6y+15z=1 \end{cases}$

b) If $6p^3+5p^2-66p+40=0$, find the possible values of p by using a non-programmable scientific calculator.

8. Given the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ where;

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 2 & -1 & 5 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 5 & -6 & 1 \end{pmatrix} \text{ and } \boldsymbol{C} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & 6 \\ 8 & 9 & 10 \end{pmatrix}$$

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a) Store the matrices A, B, and C in a non-programmable scientific calculator.

b) Determine the following:

i) $A^{-1}B^{-1}C^{-1}$ ii) $A+B-C^2$

iii) $|CA| \div |B^{-1}|$.

1.2 The computer

The computer plays a number of vital roles in the teaching and learning process. It acts as a source of materials, instrument for learning, and storage device. This section will basically describe the use of a computer as an instrument that facilitates the learning of mathematics. The main focus of the description will be on the topics of Basic Applied mathematics. Computers are installed with some basic programs which can be used to make calculations as well as graphing.

The common basic mathematical programs in any kind of computer include calculator and spread sheet. However, users can install other application programs such as Maple and Matlab which are more advanced in executing mathematical calculations and graphing.

The computer's calculator

The computer's calculator offers precise calculations and has a powerful interface. It consists of basic standard calculations with programming, scientific calculations, and statistics. Furthermore, there are also other features which are very useful like mortgage calculation, multifunctional converter, and a few more options. In this section, modes of computer's calculator will be discussed and some activities for a learner to perform will be given so as to bring understanding of the use of the computer's calculator.

Opening the computer's calculator

Follow the steps below to access the computer's calculator in Window 10 (see Figure 1.2):

Step 1: Click the Start Menu.

Step 2: Choose the calculator in the given options.

Figure 1.2 Location of computer's calculator in window 10

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In other computers with Windows 8, the calculator can be opened by typing calculator or calc in the start screen and clicking it. This is also possible for Windows 7 except that start search box is used in place of start screen.

Computer's calculation modes

Even though it has a different version in Windows 10 as compared to Windows 7, the computer's calculator is the same in both operating systems. The interface looks just the same and its features are identical.

The Computer's calculator in Windows 10 has four (4) main modes for which calculations can be done:

a) The Standard Mode

b) The Programmer Mode

c) The Scientific Mode

d) The Statistics Mode

Each mode enables the user to perform basic operations, unit conversion, time conversion, or worksheet calculations. Thus, the mode and type of operations have to be set first before calculations so that the particular kind of functional keys can be seen and used.

a) The standard mode

When you first open the computer's calculator, the standard mode will be selected by default as shown in Figure 1.3. This mode is equivalent to the normal pocket calculator. The keyboard number values, the keypad (with the number key activated) or the mouse can be used to make calculations.

Figure 1.3 Standard Mode of the computer's calculator

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b) The programmer mode

This mode gives function keys of a programmable scientific calculator. It offers the possibility to do operations with bases such as binary, octal, hexadecimal, and decimal. One can convert values from one base to another. For example, conversion from a base two number system (binary 0, 1) to a base ten number system (decimal 0 - 9). This mode also offers the logical bit operations such as XOR, OR and AND .

To access this mode, click on the option menu in the computer's calculator and select the Programmer option. (See Figure 1.4).

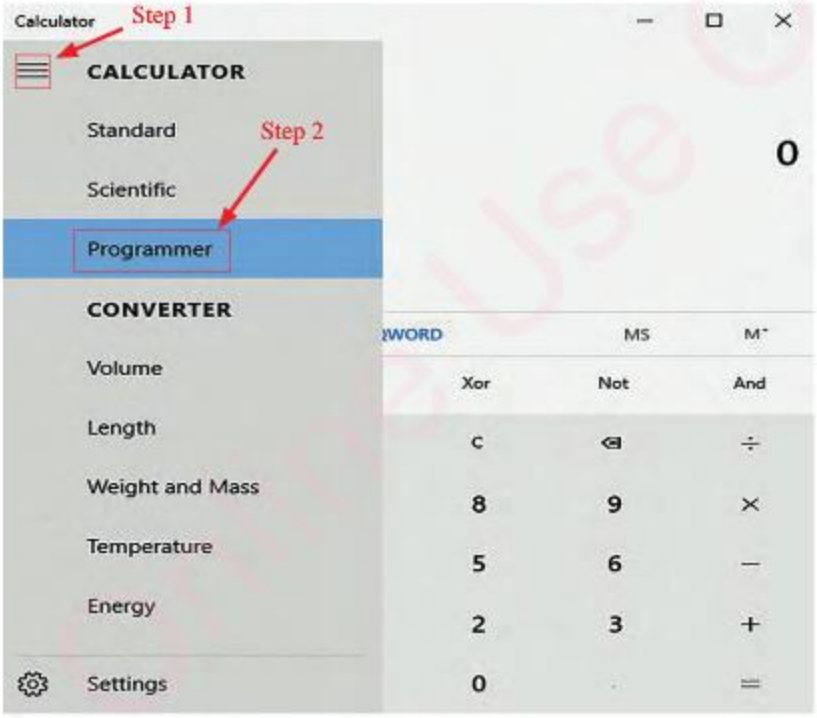

Figure 1.4 Setting the programmer Mode of the computer's Calculator

Like in non-programmable scientific calculator, the keys of this calculator are related to the base. For example, the A - F buttons are only accessible if you work in hexadecimal values. The other buttons are already common for a programmer (See Figure 1.5).

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Figure 1.5 The Programmer Mode of the Computer's Calculator

c) The scientific mode

This mode resembles a non-programmable scientific calculator. Any calculation which can be performed using hand scientific calculator may be executed. To access this mode, click on the option menu in the computer's calculator and select the scientific option. (See Figure 1.6).

Figure 1.6 Scientific Mode of the Computer's Calculator

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d) Statistics mode

Like in the standard mode of the computer's calculator, operations in statistics mode are done by clicking on a number, followed by an operation then put the second number and finish up by clicking the equal sign to obtain the final answer.

Figure 1.7 Statistics Mode of the Computer's Calculator

How to create a data set in statistics mode

The data set is a collection of data in numbers that can be used to perform different operations. To create the data set, click the number (s) which form the set of a particular data and then click **add** button. Repeat these steps until all the data are entered. Each time you press add button, the number of count changes. "Count" represents the number of entered data. The data will be displayed on the dataset list as illustrated in Figure 1.7. After all the needed data are entered in the dataset list, click on any operation such as sum, mean, or standard deviation and the corresponding result will be displayed at the bottom of the dataset list.

Remark: Clicking the **C** button in the statistics mode deletes the current data expressed before clicking the **add** button and the CAD button clears all the data from the data set.

Other options in the computer's calculator

Computer's calculator includes other tools which are used in measurement such as mass, time, volume, weight, light intensity, and distance. Thus, a computer's calculator has more things to do than the pocket or hand scientific calculator.

Exercise 1.2

1. Use a computer's calculator to determine the value of each of the following expressions.

   a) $\left(\left(\frac{13}{7}\right)^{7}\right)^{4} \div \left(\frac{1}{4} - 13\right)^{6}$

   b)
   $$\frac{(-1.76)^{5} + (3.0006)^{7}}{\left((2.12)^{-3} \times 44\right) - \left(0.0009 + \left(\frac{4}{7}\right)^{8}\right)}$$

2. Use a computer's calculator to determine the value of each of the following.

   a)
   $$\sqrt{3\tan 60^\circ} + \frac{\sqrt[4]{15}}{7}(\cos 45^\circ - \sin 120^\circ)$$

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b) $\tan^{-1}\left(\sin\left(\cos\left(\frac{\log 315}{\ln 2}\right)\right)\right)$,

compute in radian.

3. Compute the mean, sum of squares, and the standard deviation of the following data by using a computer's calculator:
12, 74, 29, 10, 70, 12, 36, 90, 10, 52, 39, 70, 12

4. Convert the following by using a computer's calculator.
   a) 60^0 into radians
   b) 100 acres into square metres
   c) 1000 cm^3 into m^3
   d) 1000 cm^3 into litres

5. Use a computer's calculator to find the difference in years, weeks, and days between:
   a) 9/12/1961 – 12/4/1962
   b) 07/7/1964 – 30/10/1995
   c) 14/9/1999 – 8/9/2017

Spreadsheet

A spreadsheet is an interactive computer application program for organization, analysis, and storage of data. The program operates on data presented as cells of an array organized in rows and columns. Each cell of the array may contain either numerical or text data.

The results from formulae can automatically be calculated and displayed in the other cell within the worksheet. Microsoft excel is one of the spreadsheet programs offered in the Microsoft Office Software Package. This program allows the user to perform calculations such as average, addition, subtraction, and finding the maximum and minimum values of numbers. In addition, Microsoft Excel can be used to create histograms, pie charts, and plot graphs of different types of functions.

Starting Microsoft excel

The steps for opening Microsoft Excel depends on the type of a computer's operating system. In this section, Microsoft excel as one of the applications (Microsoft Offices 2016) will be used as an example. For example, to open excel in Microsoft Windows 10;

1. Click the start menu
2. Select the Microsoft Office
3. Select the Microsoft Excel

Then, the excel program opens in a display similar to that shown in Figure 1.8.

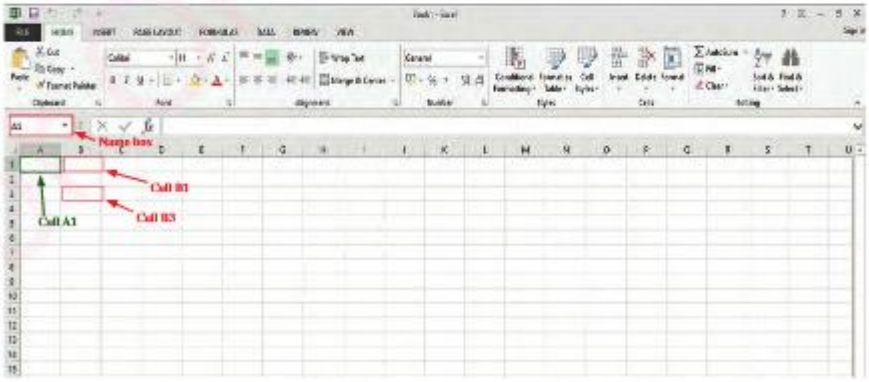

Figure 1.8 Microsoft Excel display

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The excel sheet is divided into grids called cells where the data can be entered. The cell is named by notation representing column and row. The columns are assigned capital letters of alphabets while the rows are assigned numbers. Thus, the first cell is named as A1 which means the cell is in column A and in row 1 as shown in Figure 1.8, while B1 means the cell is in another column B in the same row (row 1), and the fourth cell vertically below B1 cell is B4. Before performing any calculations, data (numbers) have to be entered to the cells. Normally, the cell name appears in the "Name box' as shown in Figure 1.8.

To enter the data, just click on the cell and type the data. The next data can be entered along the same column or row. The vertical and horizontal navigation arrows are used for moving the cursor to another cell or you may use the "mouse pointer" by clicking the required cell.

Activity 1.11

Insert the following scores in a computer's excel worksheet;

50, 10, 60, 30, 90, 47, 20, 80. 33, 89, 67, 74, 48, 96, 40, 60, 89, 67, 30

then: a) Rearrange the scores in descending order

b) Find the sum, average, maximum and minimum score.

Remark: In excel sheet, for easy operations, the data have to be entered in one column or one row. For example, the column of the data in Activity 1.11 should appear as shown in Figure. 1.9.

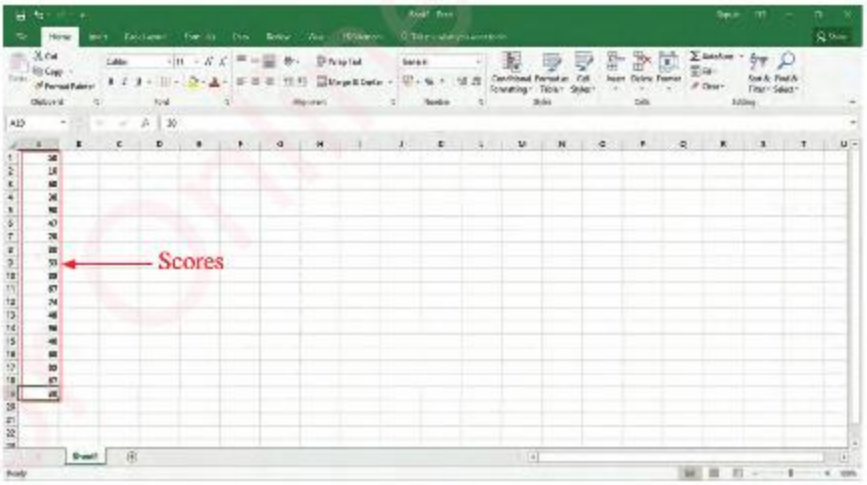

Figure 1.9 Scores of students in a certain Mathematics class

Rearrangement of data

To rearrange the data in excel, first select the data, then on home tab, click the sort button and select the type of arrangement preferred. Observe the illustration in Figure 1.10.

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The cell at which the formula is typed can be used as a reference. If the same formula applies to other data, place the cursor at the bottom of the right corner of the reference cell (the symbol "+" will appear), click and hold the cursor and drag down across all the rows. After releasing the mouse, the formula will be applied to all cells through which the dragging process took place. See Figure 1.13.

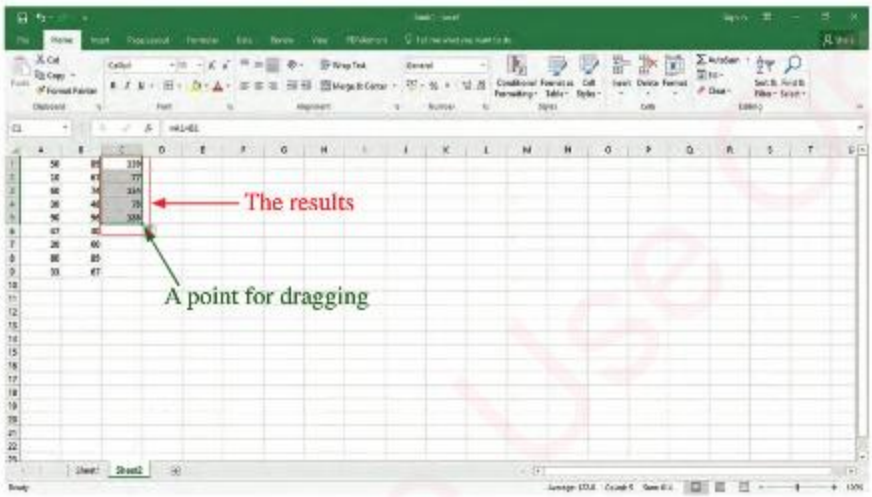

Figure 1.13 Sum of A1 and B1

Example 1.16: Use excel computer package to solve the following:

a) $(6+2)^6$

b) $(36\times 24)^2 \div (3^5 - 241)$

Solution

a) To find $(6+2)^6$, follow the steps below which are also summarized in Figure 1.14 (a and b).

Step 1: Enter 6, 2, and 6 in three consecutive excel cells (A1, B1, and C1)

Step 2: Write an excel formula using symbols and cell names $(A1+B1)^{\wedge} C1$

Step 3: Press enter to get the answer.

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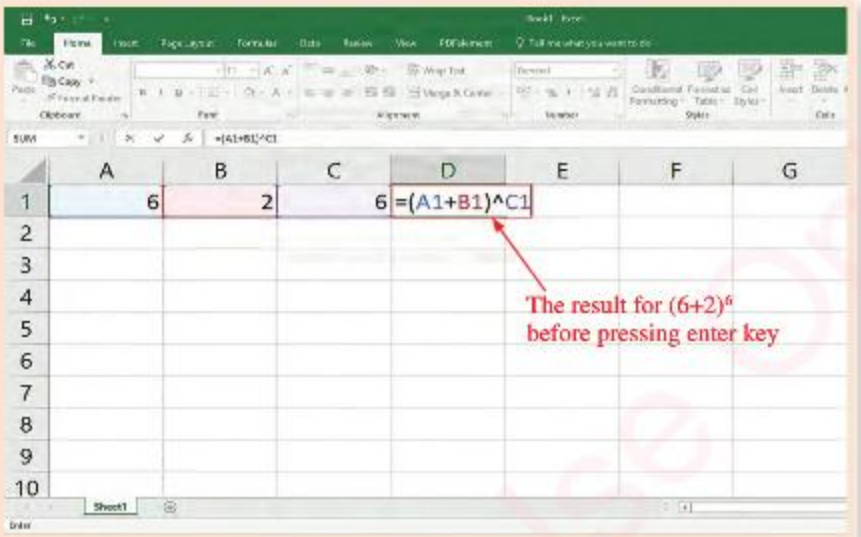

Figure 1.14 (a) Illustrating the formula in excel

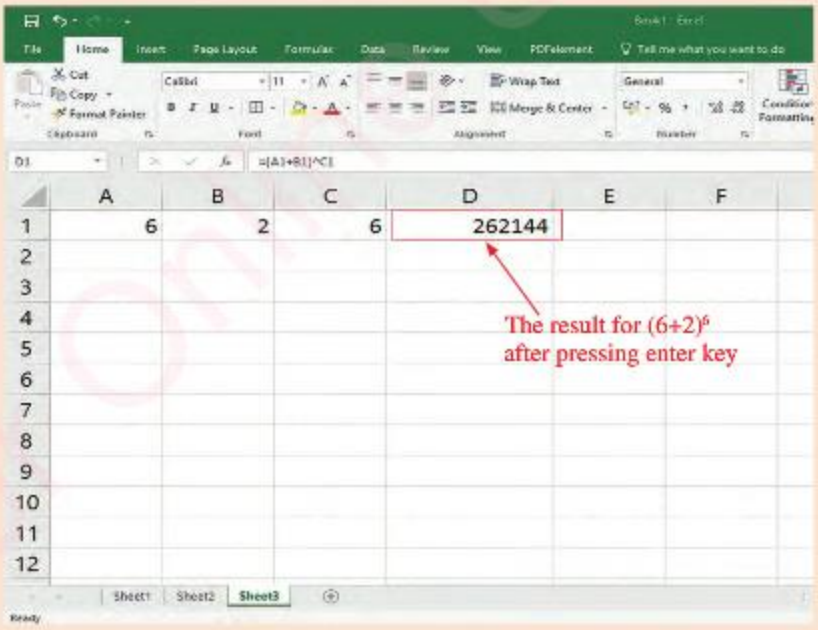

Figure1.14 (b) Illustrating the answer obtained in excel

Therefore, after following the above steps, the result is 262144 .

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b) To find $(36\times24)^2\div(3^5-241)$, do the following:

Step 1: Enter the numbers 36, 24, 2, 3, 5 and 241 in excel.

Step 2: Make an excel formula for representing the expression $(36\times24)^2\div(3^5-241)$

Step 3: Press enter.

After following the above steps, the result is 373248. See Figure 1.15 (a) and (b)

Figure: 1.15 (a) Illustrating formula in excel

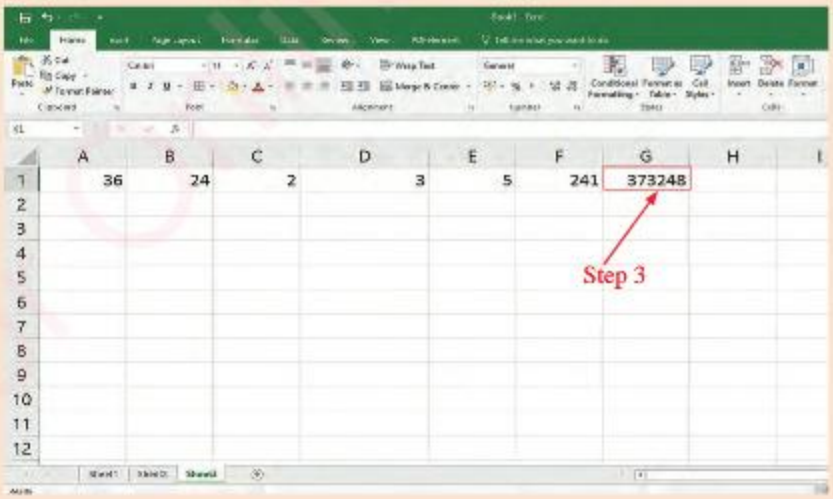

Figure: 1.15 (b) Illustrating the answer obtained in excel

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Exercise 1.3

1. Study the figure below and answer the following questions.

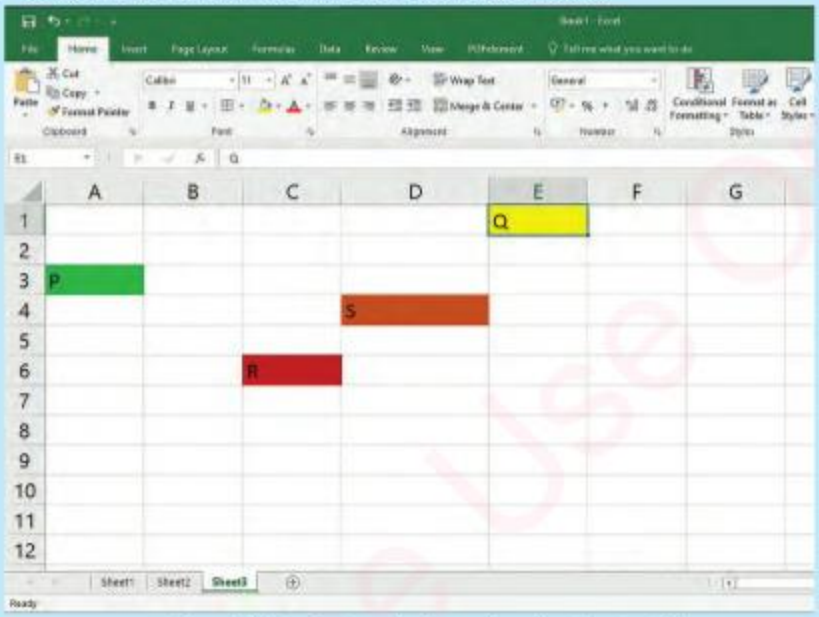

Figure 1.16 Excel sheet showing active cells P, Q, R and S

a) What does the figure represent?
b) Provide the grid name for the letters P, Q, R, and S.

2. Write the procedure for setting an excel formula in a computer to calculate $(12\times13)+(8\div2)-27$.

3. Use the Computer's excel package to solve the following:

a) $\dfrac{2(23+11)}{(7-10)\div0.9}$ b) $(12\times13)-(4\div8)+270$

c) $\sqrt{21.8}+\sqrt[4]{781}$ d) $\left(1+(0.00031)^6\right)^2$

4. The table below presents the scores of 10 best candidates in five subjects.
   a) Write the procedure for finding the total and mean of the scores using the spreadsheet package.
   b) Determine the total and mean score of each student.
   c) Rearrange the scores in descending order of total score and then state the best three candidates.

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Table 1.6 Scores for 10 best candidates in five subjects

S/N	Student Name	BAM	Physics	Chemistry	Biology	G.S
1	Candidate 1	80	77	63	90	78
2	Candidate 2	40	94	44	78	90
3	Candidate 3	55	79	98	57	69
4	Candidate 4	77	96	55	88	65
5	Candidate 5	49	88	81	50	33
6	Candidate 6	73	98	59	85	55
7	Candidate 7	45	71	94	70	88
8	Candidate 8	92	43	77	97	72
9	Candidate 9	68	39	68	99	56
10	Candidate 10	88	29	69	64	91

5. Complete the following table by using an excel spreadsheet:

Table 1.7 Data representing variables x and y

x	y	x^2	y^2	$(xy)^2$	x^2y^2
11	101				
21	91				
31	81				
41	71				
51	61				
$\sum x =$	$\sum y =$	$\sum x^2 =$	$\sum y^2 =$	$\sum (xy)^2 =$	$\sum (x^2y^2) =$

6. Use a computer spreadsheet to find the mean, median, mode, variance, and standard deviation of the following data:
3, 13, 63, 43, 23, 83, 53, 33, 93, 73, 13, 53, 73, 15, 78, 45, 73, and 63.

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Revision exercise 1

1. Use a non-programmable scientific calculator to compute each of the following expressions:

   a) $\dfrac{(0.6284)^2(62.45)^3(142.72)^4}{\sqrt{(12.68)^3}(150.76)^5}$

   b) $\dfrac{\sin 62° \times \cos 43.5° \times e^2}{10^{-2} \times \log 140.5 \times \tan 75.2°}$

   c) $\dfrac{{}^{12}C_8 \times {}^{10}P_2 \times \sin^{-1}(0.8695)}{\sqrt[3]{812.5} \times \tan^{-1}(1.5) \times \ln(18.62)}$

2. Use a non-programmable scientific calculator to evaluate each of the following:

   a) $\dfrac{d}{dx}\left(x^4+4x^3+2x^2-10x+6\right)$, when $x=0.5654$

   b) $\dfrac{d}{dx}\sqrt{1+4x+x^2}$, when $x=6$

3. Use a non-programmable scientific calculator to evaluate each of the following:

   a) $\int_0^1 (2x+1)e^{x^{3+x+1}}dx$

   b) $\int_{-2}^{3}\left(2x^3+8x^2-6x-16\right)dx$

   c) $\dfrac{d}{dx}\left(x^4+4x^3+2x^2-10x+6\right)$ when $x=6$

4. Use a non-programmable scientific calculator to solve each of the following systems of simultaneous equations:

   a) $\begin{cases} 5x+6y=450 \\ 2x+2y=160 \end{cases}$ b) $\begin{cases} 6x+6y=900 \\ 12x+8y=1440 \end{cases}$

5. Use a non-programmable scientific calculator to solve each of the following systems of simultaneous equations:

   a) $\begin{cases} 3x+3y+3z=21 \\ 5x+10y+15y=80 \\ 7x+21y+28z=154 \end{cases}$ b) $\begin{cases} 5x+10y-15z=-20 \\ 8x+12y+8y=8 \\ 24x-24y-32z=88 \end{cases}$

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6. Use a non-programmable scientific calculator to find the solution of each of the following equations:

   a) $24x^2-38x+15=0$ b) $x^2+9x-400=0$

   c) $x^3-5x^2-102x+216=0$ d) $30x^3-43x^2-7x+6=0$

7. Insert the following statistical data in a non-programmable scientific calculator and answer the quetions that follow:

Table 1.8 Statistical data

x	15.5	25.5	35.5	45.5	55.5	65.5	75.5	85.5	95.5
f	25	10	8	12	48	4	26	34	33

Find:

a) The mean b) The standard deviation c) $\sum x$ d) $\sum x^2$

8. Complete the following table by using Excel spreadsheet:

Table 1.9 Data representing variables x and y

x	8	12	24	16	10	30	25	5	$\sum x =$
y	16	18	20	14	12	22	32	48	$\sum \mathbf{y} =$
x^2									$\sum x^2 =$
y^2									$\sum \mathbf{y}^2 =$
$(\mathbf{xy})^2$									$\sum (\mathbf{xy})^2 =$
$\mathbf{x}^2\mathbf{y}^2$									$\sum (\mathbf{x}^2\mathbf{y}^2) =$

9. Using Excel spreadsheet, complete the following table of scores for ten best candidates in *HGE* combination at a certain school.

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Table 1.10 Scores for 10 best candidates in HGE Combination at a certain school

S/N	History	Geography	Economics	GS	Bam	Total	Average	Rank
1	70	72	68	81	46			
2	65	52	67	35	30			
3	54	48	72	69	15			
4	82	88	80	64	75			
5	76	75	62	51	40			
6	50	86	47	38	42			
7	38	54	60	73	21			
8	45	61	35	64	25			
9	86	92	70	65	83			
10	72	60	57	85	31			

10. The length (in cm) of 15 pieces of ropes are given as follows: 38, 52, 80.5, 75.5, 47, 68.5, 16.7, 29.8, 37.5, 80.6, 78.2, 65.9, 18.7, 29.7, 83.2. Compute the following using Spreadsheet.

    a) Mean b) The variance c) The Standard deviation (δ_x)

    d) $\sum x^2$ e) $\sum x$

11. Use Spreadsheet to compute each of the following expressions:

    a) $\dfrac{\sqrt{64.3}+\sqrt[5]{162.8}-(2.68)^3}{(71.6-19.9)\div 0.58}$ b) $(46\times 95)-(16\div 0.05)+(292-12)$

    c) $\left(16.59+(0.00045)^7\right)^4$ d) $\dfrac{(72.5\times 12.8)^6\div\left(8^4+267\right)}{\left(6^4\times 4^3\right)-\left(19^2+2^3\right)}$

12. Use a computer's calculator to determine the value of each of the following expressions.

    a) $\sqrt[3]{6\sin 32.5^\circ}+\sqrt[4]{\cos 82.5^\circ}-\dfrac{1}{3}\sqrt{12.8}$

    b) $\tan^{-1}\left[(0.6875)^2+(1.028)^2\right]$, give the answer in radians.

    c) The mean and standard deviation of the data: 126, 158, 162, 145, 136, 192, 181, 175, 168, 170, 140, 160, 154, 164, 170, 161, 191, 194, 141, 132.

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13. Given matrices $A = \begin{pmatrix} 8 & 12 & 14 \\ 5 & 2 & 3 \\ 6 & 7 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & -4 & 10 \\ 9 & -5 & -7 \\ 1 & 13 & -3 \end{pmatrix}$. Use a non-programmable scientific calculator to find: a) AB b) $A^{-1} + B^{-1}$ c) $(AB)^T$ d) $|AB|$

14. Given the frequency distribution table below:

Table 1.11 Frequency distribution table for grouped data

Class interval	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100
Frequency	25	14	12	20	28	16	39	15	31

By using a non-programmable scientific calculator find:

a) The mean b) The standard deviation c) The variance d) $\sum x^2$

15. Using a non-programmable scientific calculator make the following conversions:

a) 350°F into degree of Celsius.

b) 2500 m into feet.

c) 400 miles to kilometres.

d)160 m / s into km / h

e) 2550 acres into square metres.

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CHAPTER
TWO

Functions

Introduction

In Mathematics, the description of relationship among sets of things or quantities is emphasized in the topic of relations and functions. Relations describe all types of dependency, while functions are based on the description of relations with special characteristics. The knowledge of functions is important especially when one wants to describe mathematically the relationship that exists between two or more parameters. It is used, for example, in designing machines, predicting natural disasters, studying the trend of diseases, understanding world economies, and keeping aeroplane in air. In this chapter, you will learn about the types and properties of functions, sketch graphs of polynomial functions, step functions, rational functions, their features as well as their applications.

2.1 The concept of a function

The concept of function can be better explained by referring the relationship between the set of children and set of fathers. When the set of children is mapped onto that of fathers it is expected that there will be no more than one arrow drawn from a single element in the set of children to the set of fathers but the vice versa is true. See Figure 2.1 which shows the children – father relationship as an example of function.

Thus, a function f is defined as a rule that assign each element x in set A to exactly one element $f(x)$ in set B whereby set A is called domain of f and B is called range of f.

A function can take input from many variables but always give the same unique output to that function.

Any set of inputs in a function is called **domain** while a set of output of a function is called **range.**

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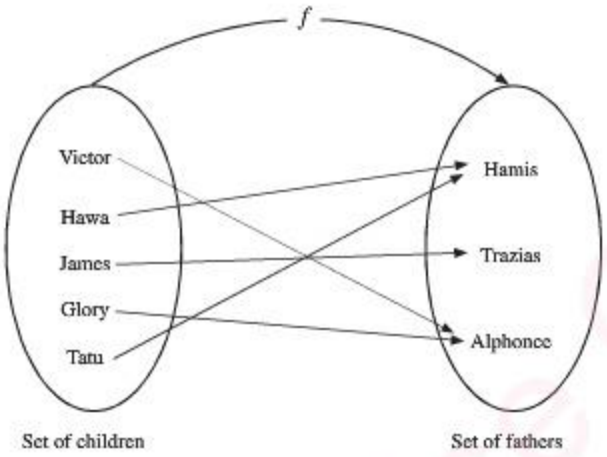

Figure 2.1 A pictorial diagram for the function f

Based on the description above and in the illustration in Figure 2.1, a function can be defined as a mapping of two sets related in either on one to one or many to one correspondence.

Function notation

Mathematical notation of a function can be written as $f: A \to B$, where f represents the term function, A is a set of input (domain), and B is a set of output (range) of f. The arrow implies that elements in set A are mapped onto that of B. The function $f: A \to B$ is read as "f is a function such that elements in set A are mapped onto set B".

Consider the function f which maps x onto $5x+1$. Using mathematical notation, this can be represented by the notation; $f: x \to 5x+1$ or $f(x)=5x+1$. This is read as; "f is the function such that x is mapped onto $5x+1$ or the function of x equals to $5x+1$" or "x mapps onto $5x+1$".

Example 2.1:

a) If b is an input element of a function f and $f(b)$ is an output element of the function; write the expression for the function notation.

b) Given that $y=x^2$, write down the rule governing this relation.

c) If the function f is such that $f(x)=\sqrt{x}$ and $x=a$, write down the corresponding ordered pair in terms of a.

Solution

a) Function notation, $f: b \to f(b)$

b) The rule governing the given relation is $f: x \to x^2$ or $f(x)=x^2$

c) The rule is $f: x \to \sqrt{x}$ which gives $\mathbf{f}: \mathbf{a} \to \sqrt{\mathbf{a}}$. Therefore, the ordered pair is $\left(\mathbf{a}, \sqrt{\mathbf{a}}\right)$

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Pictorial representation of a function

Pictorial representation of a function is a representation of a function using diagrams. It involves mapping of elements from the first set to the elements of the second set.

Let $A=\{a, b, c, d\}$ and $B=\{e, f, g, h\}$: the pictorial representation of these sets is presented in Figure 2.2.

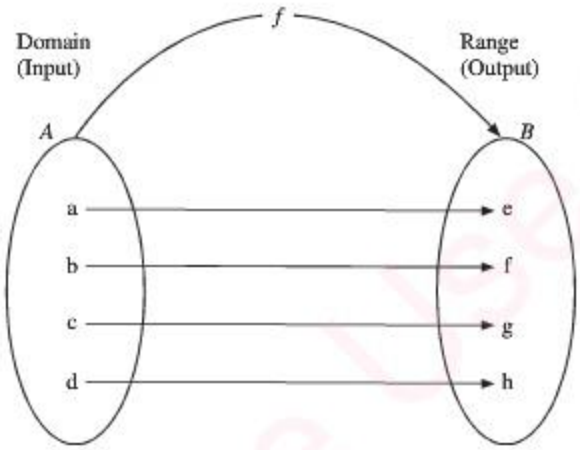

Figure: 2.2 The pictorial representation of sets

Example 2.2: Let set $P=\{2, 3, 5, 7\}$ be the input of the function defined by $f(x)=2x$. Represent this function pictorially.

Solution

Given $f: x \rightarrow 2x$ which can also be expressed as $f(x) = 2x$, substitute each value of set P in the function $f(x) = 2x$ to obtain another set, say L as follows:

Table 2.1 Table of values for $f(x)=2x$

Set P (x value)	2	3	5	7
Set L ($f(x)$ value)	4	6	10	14

The pictorial representation is as follows:

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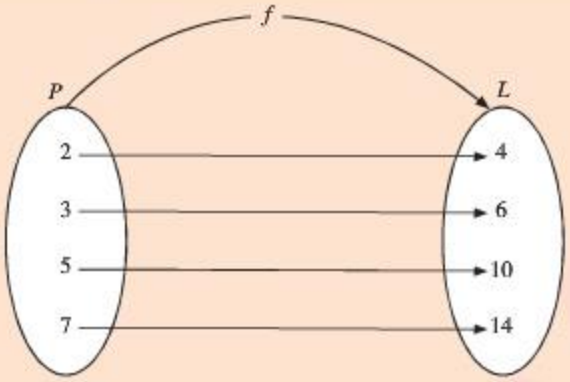

Figure 2.3 The pictorial representation of set P onto L

Exercise 2.1

1. For each of the following set of ordered pairs:

   a) $\{(3,3), (4,2), (5,1), (6,0)\}$

   b) $\{(2,0), (4,8), (2,1), (3,6)\}$

   c) $\{(1,1), (2,2), (3,1), (1,6)\}$

   i) Identify the domain and range.
   ii) Represent the ordered pairs pictorially.
   iii) Identify whether each of the ordered pair is a function.

2. Consider the pictorial diagram below.

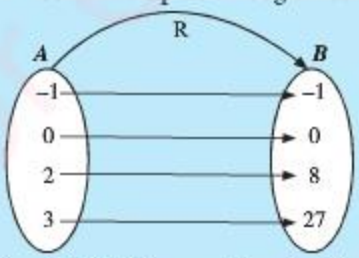

Figure 2.4 Pictorial representation of set A and B.

   a) State the domain and range
   b) Write the elements as a set of ordered pairs

3. If set $A=\{0,1,2,3,4\}$ is the domain of the function f defined by $f(x)=x^2+3$
   a) Find the range
   b) Write the elements as a set of ordered pairs
   c) Represent the given function pictorially

4. State, with reasons, whether each of the following mappings represents a function or not.

   a)

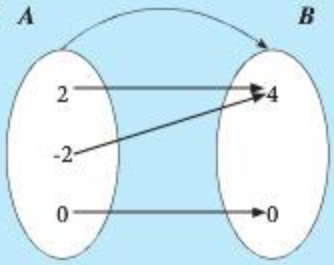

Figure 2.5 Pictorial representation of set A and B.

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b)

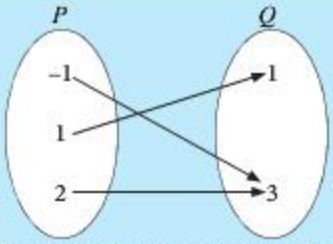

Figure 2.6 Pictorial representation of set P and Q.

c)

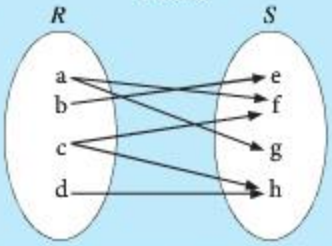

Figure 2.7 Pictorial representation of set R and S.

d)

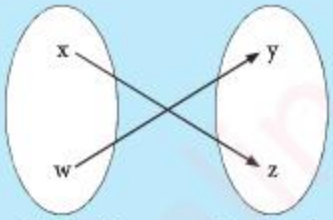

Figure 2.8 Pictorial representation of set U and V.

5. Amina is a mother of Bakari and Saada while Agnes is a mother of John, Neema and Fadhili. Use the following relation to represent the set of domain and range pictorially.

$R=\{(x,y): y \text{ is a mother of child } x\}$

Types of functions

Functions are classified as either algebraic or transcendental functions. The algebraic functions include polynomial and rational functions while transcendental functions include exponential, logarithmic, and trigonometric functions. The focus of this section is on polynomial functions, rational, and step functions.

a) Polynomial functions

Polynomial functions are functions in the form $P(x)=a_n x^n + a_{n-1}x^{n-1} + ..a_1 x + a_0$, where $a_n, a_{n-1}, ..., a_1, a_0$ are real numbers called the coefficients of the polynomial, $a_n \neq 0$ and $\mathbf{n}$ is non-negative integer. The highest exponent is the degree of a polynomial. The description of the types of polynomials is given in Table 2.2.

Table 2.2 Polynomial Functions

S/N	General polynomial	Degree of polynomial	Type of the polynomial	Example
1	$\mathbf{f(x)=a_0x^0=a_0}$	0	constant function	$\mathbf{f(x)=-2}$
2	$\mathbf{f(x)=a_1x+a_0}$	1	linear function	$\mathbf{f(x)=3x+5}$
3	$\mathbf{f(x)=a_2x^2+a_1x+a_0}$	2	quadratic function	$\mathbf{f(x)=4x^2-7x+5}$
4	$\mathbf{P(x)=a_3x^3+a_2x^2+a_1x+a_0}$	3	cubic function	$\mathbf{f(x)=3x^3-x^2+6x+10}$

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Example 2.3: For each of the following functions, identify whether it is a polynomial or not. For polynomial functions, give the name and state its degree.

i) $f(x)=2x^3-5x^2+4x-1$

ii) $f(x)=3x^{12}+\frac{2}{x}$

iii) $f(x)=x+2$

iv) $f(x)=0$

v) $f(x)=3x^3+2x+\sqrt{x}$

Solution

i) $f(x)=2x^3-5x^2+4x-1$ is a cubic polynomial of degree 3.

ii) $f(x)=3x^{12}+\frac{2}{x}$ is not a polynomial because of $\frac{2}{x}=2x^{-1}$ (-1 is a negative integer)

iii) $f(x)=x+2$ is a linear polynomial of degree 1.

iv) $f(x)=0$ is a constant polynomial of degree 0.

v) $f(x)=3x^3+2x+\sqrt{x}$ is not a polynomial because of $\sqrt{x}=x^{\frac{1}{2}}$ ($\frac{1}{2}$ is not a positive integer)

b) Rational functions

Rational function f is a function of the form:

$$f(x)=\frac{P(x)}{Q(x)};\ \text{where } Q(x)\neq 0$$

in which P and Q are the polynomials. The domain consists of all values of x such that $Q(x)\neq 0$. The function, $f(x)=\frac{1}{x}$ is an example of a rational function.

c) Step functions

Step functions are functions whose variables have a special relationship under certain conditions that make the graph to break in intervals. These functions are expressed stepwise. For instance, the graph of the function $f(x)=\begin{cases}2 & \text{if } x<0\\ -1 & \text{if } x\geq 0\end{cases}$ is shown in Figure 2.9.

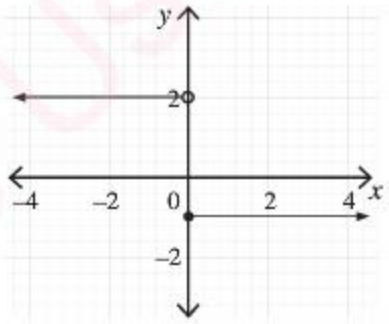

Figure 2.9 An example of a graph of a step function

Operations on functions

The four basic algebraic operations which are addition, subtraction, multiplication, and division can be applied to functions. Operations on two or more functions, the result is a function.

a) Addition

Addition of functions involves adding the coefficients of like terms whose variables have the same exponents.

Let $P(x)$ and $Q(x)$ be polynomial functions, then $P(x)+Q(x)=(P+Q)(x)$.

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Example 2.4. Find $(P+Q)(x)$ if;

$P(x)=3x^3+2x^2+x+1$ and $Q(x)=x^3+x^2$

Solution

From

$P(x)=3x^3+2x^2+x+1$ and $Q(x)=x^3+x^2$

$(P+Q)(x)=\left(3x^3+x^3\right)+\left(2x^2+x^2\right)+x+1$

$\therefore (P+Q)(x)=4x^3+3x^2+x+1$

Example 2.5: Find $f(x)+g(x)$, given

$$f(x)=\frac{2}{x-3} \text{ and } g(x)=\frac{4}{x+1}$$

Solution

From $f(x)+g(x)=\frac{2}{x-3}+\frac{4}{x+1}$

$$=\frac{2(x+1)+4(x-3)}{(x-3)(x+1)}$$

$$=\frac{2x+2+4x-12}{(x-3)(x+1)}$$

$$\therefore f(x)+g(x)=\frac{6x-10}{x^2-2x-3}$$

b) Subtraction

Subtraction of functions involves subtracting the coefficients of terms whose variables have the same exponents.

Let $P(x)$ and $Q(x)$ be polynomial functions. Then, $P(x)-Q(x)=(P-Q)(x)$.

Example 2.6: Given $P(x)=4x^3+x^2+1$ and $Q(x)=x^3-5$, find $(P-Q)(x)$.

Solution

From

$$(P-Q)(x)=\left(4x^3+x^2+1\right)-\left(x^3-5\right)$$

$$=4x^3+x^2+1-x^3+5$$

$$=4x^3-x^3+x^2+1+5$$

$\therefore (P-Q)(x)=3x^3+x^2+6$

Exercise 2.2

1. Find $(x)+(x)$ for each of the following pairs of functions and

   a) $(x)=2x+3$ and $(\ )=\ ^2+7\ -1$

   b) $(x)=6x^2+7x+3$ and $(x)=4x^3+x+9$

2. Find $g(x)-f(x)$ for each of the following pairs of functions and

   a) $(x)=10x+3$ and $(\ )=13\ -7$

   b) $(x)=\frac{3}{x+2}$ and $(x)=\frac{9}{x+2}$

3. For each of the following pairs of functions and , find $f(x)+g(x)$. Hence, state the type and the degree of the resulting polynomial.

   a) $(\ )=\ ^3-2\ ^2+11$ and $(x)=3x^3+6x+7$

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b) $f(x)=\frac{7}{2x+1}$ and $g(x)=\frac{1}{x-1}$

4. For each of the following pair of functions f and g, find $f(x)-g(x)$ and hence state the type and degree of the resulting polynomial.

a) $f(x)=2x^2+3x+1$ and

$g(x)=2x^2+5x+3$

b) $f(x)=2x^3-x+5$ and

$g(x)=7x^3+3x+11$

5. If $f(x)+g(x)=\frac{6x+10}{x^2+x-12}$ and

$f(x)=\frac{2}{x+4}$, show that $g(x)=\frac{4}{x-3}$

c) Multiplication

The product of two polynomial functions f and g is denoted by $fg(x)$ where $fg(x)=f(x)g(x)$. In order to multiply $f(x)$ and $g(x)$, all possible products of a term in $f(x)$ and a term in $g(x)$ are formed and then the sum of all such products is found.

If $f(x)$ has degree m and $g(x)$ has degree n then $f(x)\times g(x)$ has degree $m+n$.

Let $f(x)$ and $g(x)$ be polynomial functions, then $f(x)\times g(x)=(fg)(x)$.

Example 2.7: Given $f(x)=x^2+3x+1$ and $g(x)=x+1$, find $f(x)\times g(x)$.

Solution

From $f(x)\times g(x)=(fg)(x)$

$=(x^2+3x+1)(x+1)$

$=x^3+x^2+3x^2+3x+x+1$

$\therefore f(x)\times g(x)=x^3+4x^2+4x+1$

Exercise 2.3

1. If $g(x)=2x+3$ and $h(x)=x^2+4x-1$, find $g(x)h(x)$

2. Given $f(x)=x+2$, $g(x)=x-1$ and $h(x)=x+3$, find;

   a) $f(x)g(x)h(x)$

   b) $g(x)h(x)$

3. If $h(t)=2t^2-3t+1$ and $g(t)=3t+2$, find $h(t)g(t)$.

4. Find $f(x)g(x)$, given that

$f(x)=\frac{2}{x+3}$ and $g(x)=\frac{5}{x+3}$.

5. Find the product of the functions $f(x)=x^3-8$ and $g(x)=x^2-2x+1$. Hence, state the type of the resulting polynomial and its degree.

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d) Division

The division on polynomial functions is mostly done by using long division method similar to division of numbers.

Let $P(x)$ and $D(x)$ be polynomial functions, $Q(x)$ be the quotient, and r be the remainder.

So, $\frac{P(x)}{D(x)} = Q(x) + \frac{r}{D(x)}$, where $P(x)$ is called a dividend and $D(x)$ the divisor.

Make $P(x)$ the subject. Hence ;

$$P(x) = Q(x)D(x) + r$$

Example 2.8:

If $P(x) = x^3 - 3x^2 + 4x + 2$ and $D(x) = x + 1$. Find $\frac{P(x)}{D(x)}$.

Solution

$\frac{P(x)}{D(x)}$ is done using long division method

Now, $\frac{P(x)}{D(x)} = \frac{x^3 - 3x^2 + 4x + 8}{x+1}$

$$\begin{array}{r} x^2 - 4x + 8 \\ x+1 \overline{)\, x^3 - 3x^2 + 4x + 2} \\ -\{x^3 + x^2 \\ \hline -4x^2 + 4x \\ -\{-4x^2 - 4x \\ \hline 8x + 2 \\ -\{8x + 8 \\ \hline -6 \end{array}$$

$$\therefore \frac{P(x)}{D(x)} = x^2 - 4x + 8 - \left(\frac{6}{x+1}\right)$$

Example 2.9

Given $P(x) = x^3 - x^2 - 4x + 4$ and $D(x) = x - 1$, find $\frac{P(x)}{D(x)}$ and factorize $P(x)$ completely.

Solution

$\frac{P(x)}{D(x)}$ is done by using long division

So, $\frac{P(x)}{D(x)} = \frac{x^3 - x^2 - 4x + 4}{x-1}$

$$\begin{array}{r} x^2 - 4 \\ x-1 \overline{)\, x^3 - x^2 - 4x + 4} \\ -\{x^3 - x^2 \\ \hline -4x + 4 \\ -\{-4x + 4 \\ \hline 0 \end{array}$$

$$\therefore \frac{P(x)}{D(x)} = x^2 - 4$$

Factorizing $P(x)$ from the above result gives, $P(x) = D(x)\left(x^2 - 4\right)$

but, $D(x) = x - 1$

Thus, $P(x) = (x-1)(x-2)(x+1)$

$\therefore P(x) = (x-1)(x-2)(x+1)$

Exercise 2.4

1. Find $\frac{P(x)}{D(x)}$ if $P(x) = 3x + 1$ and $D(x) = x - 1$

2. Given that $f(x) = 2x^2 + 3x + 1$ and

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$g(x)=2x+1$,find the quotient, $Q(x)=\dfrac{f(x)}{g(x)}$.

3. If $f(x)g(x)=x^4+8x^3+10x^2-x+6$ and $g(x)=x+2$, find $f(x)$.

4. Find $\dfrac{P(x)}{Q(x)}$ given that $P(x)=2x^3+7x^2-x+3$ and $Q(x)=x-1$, then state the remainder.

5. Divide $P(x)=x^3-3x^2-x+3$ by $D(x)=x-3$ and hence find all factors of $P(x)$.

2.2 Graphs of functions

Graph is one of the geometrical ways of visualising and studying the behaviour of any function and is a widely used method to represent functions. Suppose $y=f(x)$ is defined by the set of paired values (x, y). Using $xy-$plane, the function can be plotted to produce a graph that describes the geometrical relation.

A curve usually explains a certain relationship. It was noted earlier that not all relations are functions. Likewise, not all graphs represent a function. To determine whether a graph is a function, the vertical line can be drawn to test it. Since a function is a relation where there is only one value of dependent variable for each independent variable; then, if the line is drawn parallel to y-axis, the line shall cross a graph of a function at only one point. Otherwise, it is not a function.

Activity 2.1

1. Why is it necessary to represent functions graphically?
2. Which of the following graphs in Figure 2.10 are functions?

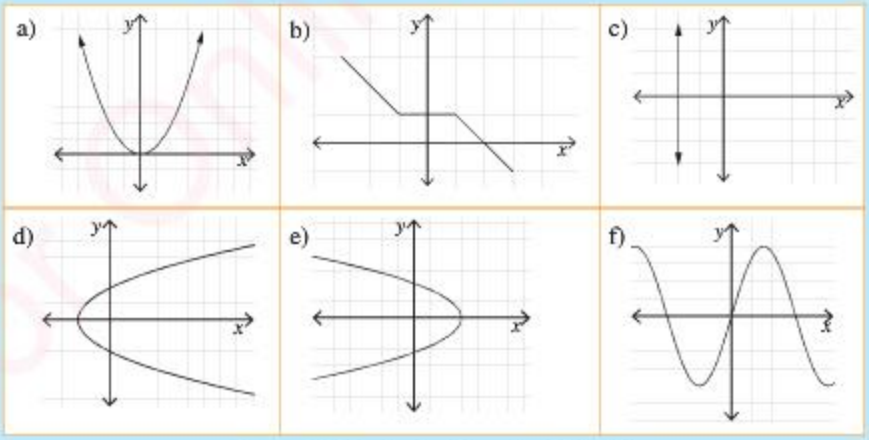

Figure 2.10 Graphs of relations for Activity 2.1

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a) Constant functions

A constant function is a function of the form $f(x)=c$, where c is a constant. The constant function can be presented graphically. The slope of a constant function is zero for a horizontal function since change in y is zero. Examples of constant functions are $f(x)=3$, and $g(x)=10$.

Example 2.10: Draw the graph of $f(x)=4$.

Solution: To be able to draw the graph of the given function, prepare a table of values as shown in the Table 2.3. The resulting graph is presented in Figure 2.11.

Table 2.3 Table of values for $f(x)=4$

x	–3	–2	–1	0	1	2	3
$f(x)=4$	4	4	4	4	4	4	4

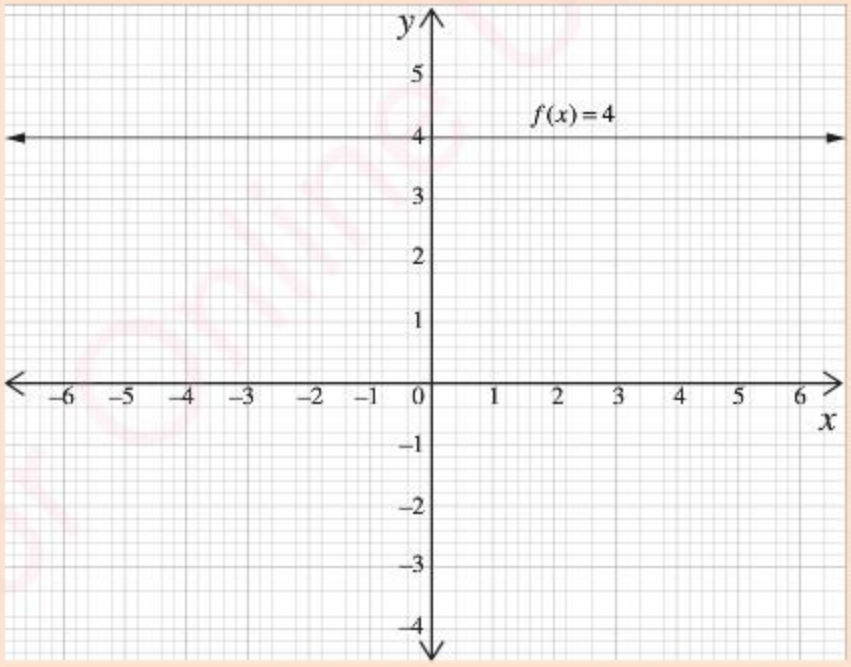

Figure 2.11 Graph of $f(x)=4$

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b) *Linear functions*

A linear function is a function of the form $f(x)=ax+b$ where a and b are constants and $a\neq 0$. The function $f(x)=ax+b$ can also be written as $y=ax+b$, where $y=f(x)$. Any equation in the form of $y=mx+c$ is said to be an equation of a straight line where m is the gradient or slope and c is the value of y when $x=0$ or $y-$intercept.

Hence, $f(x)=ax+b$ geometrically represents a straight line whose slope is $m=a$ and crosses the $y-$axis at point $(0, b)$.

The graph of $f(x)=ax+b$ can be drawn using:

i) table of values for x and y

ii) x and $y-$intercepts

By using table of values

Example 2.11: Draw the graph of $f(x)=2x+3$.

Solution

Table 2.4 Table of values for $f(x)=2x+3$

x	–2	–1	0	1	2
$f(x)$	–1	1	3	5	7

Plotting the values in $x-y$ plane, Figure 2.12 is obtained.

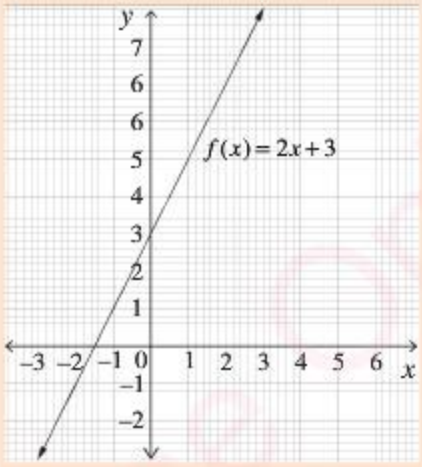

Figure 2.12 Graph of $f(x)=2x+3$

By using x and y–intercepts

Linear graphs can be drawn by using x and y intercepts. The $y-$intercept is defined as the value of y when $x=0$ or is the point where a graph crosses the $y-$axis; and $x-$intercept is defined as the value of x when $y=0$ or is the point where the graph crosses the $x-$axis. From Example 2.11, $y-$intercept is at $(0, 3)$ and $x-$intercept is at $\left(\frac{-3}{2}, 0\right)$.

Example 2.12: Draw the graph of $f(x)=-3x+3$ using x and y intercepts. From the graph, state the domain and range.

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Solution

Table 2.5 Table of values for x and y intercepts

x	0	1
y	3	0

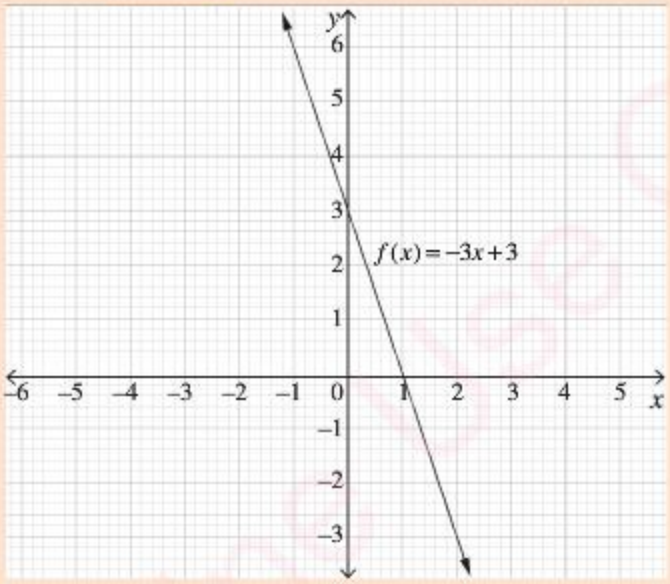

Figure 2.13 Graph of $f(x) = -3x + 3$

From the graph, a straight line drawn extends in both directions of x and y axis and no restriction given to the domain. So the domain and range are set of all real numbers.

Exercise 2.5

1. Draw the graph of $f(x) = 3$.
2. Use the table of values to draw the graph of $y = \frac{2}{3}x + 4$
3. Use x and y intercepts to draw the graph of $y = 2x + 1$.
4. Draw the graph of $f(x) = -2$ and use the graph drawn to determine the domain and range of the $f(x)$.
5. Draw the graph of $(\ ) = -3\ +1$, where $-1 < \ \leq 3$ hence state the domain and range of $(\ )$.

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c) *Quadratic functions*

A quadratic function is a polynomial function of degree two, and therefore it can be written in the form $f(x)=ax^2+bx+c$ where a, b, and c are real numbers and $a\neq 0$.

The graph of any quadratic function is a parabola. It either opens upward when a is positive $(a>0)$ or downwards when a is negative $(a<0)$. See Figures 2.14(a) and (b).

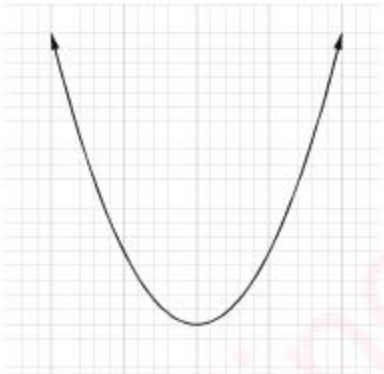

Figure 2.14 (a) The curve opens upwards when $a>0$

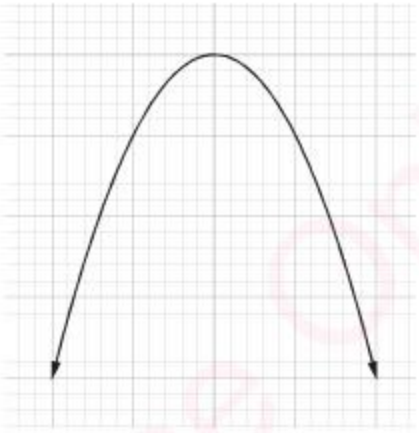

Figure. 2.14 (b) The curve opens downwards when $a<0$

Graphs of quadratic functions can be drawn or sketched by using;

i. table of values

ii. turning points and intercepts

i. Using table of values

Example 2.13 Draw the graph of $f(x)=x^2-2x-3$

Solution

Table 2.6 Table of values for $f(x)=x^2-2x-3$

x	–2	–1	0	1	2	3	4
y	5	0	–3	–4	–3	0	5

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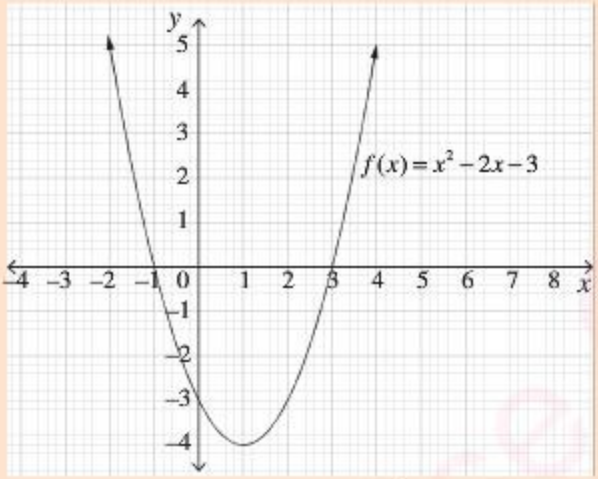

Figure 2.15 Graph of $f(x)=x^2-2x-3$

Activity 2.2

Given the following:

a) $f(x)=(x-3)^2$, $g(x)=2$, $h(x)=(f+g)(x)$

b) $f(x)=2x^2+4$, $g(x)=3x-1$, $h(x)=(f-g)(x)$

c) $f(x)=x+6$, $g(x)=4x-1$, $h(x)=(f\times g)(x)$

d) $f(x)=x^2+x-6$, $g(x)=x-2$, $h(x)=\left(\frac{f}{g}\right)(x)$

In groups of students, attempt the following:

i) Determine $h(x)$ in each case.

ii) Draw the graph of $h(x)$ in each case.

iii) Determine the domain and range of $h(x)$ in each case.

ii. Using turning points and intercepts

The turning points are the points where the curve bends (or turns). A quadratic function has one turning point. This turning point can be either the lowest point (minimum) or the highest point (maximum) on the entire curve. The minimum point occurs when the curve opens upward (i.e. when the coefficient of x^2 is positive) while the maximum point occurs when the curve opens downward (i.e. when the coefficient of x^2 is negative). See Figure. 2.14 (a and b). When the turning points and intercepts are identified, it can be used to sketch the graphs of quadratic functions.

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Determination of the turning points

Consider the quadratic function $f(x)=ax^2+bx+c$

By completing the square, we have

$$f(x)=ax^2+bx+c=a\left(x^2+\frac{b}{a}x\right)+c$$

$$=a\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right]+c-a\left(\frac{b}{2a}\right)^2$$

$$=a\left[x^2+\frac{b}{a}x+\left(\frac{b^2}{4a^2}\right)\right]+c-a\left(\frac{b^2}{4a^2}\right)$$

$$=\left[\left((x^2+\frac{b}{a}x)+\frac{b^2}{4a^2}\right)+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

$$=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$$

Let, $h=\frac{b}{2a}$ and $k=\frac{4ac-b^2}{4a}$

Then, $f(x)=a(x+h)^2+k$

Using the equation $f(x)=a(x+h)^2+k$, the turning point can be identified as $(-h, k)$. If a is positive, the turning point is a minimum point and if a is negative, the turning point is a maximum point.

The value of x at the turning point is obtained by taking $x+h=0$, so that $x=-h$; where h is a positive real number. Also, from;

$f(x)=a(x+h)^2+k,$

$y=a(0)^2+k,$ hence;

$y=k$

The line $x=-h$ is called the line of symmetry because it divides the curve (parabola) into two equal parts. Figure. 2.16 (a and b) shows the turning points and the line of symmetry.

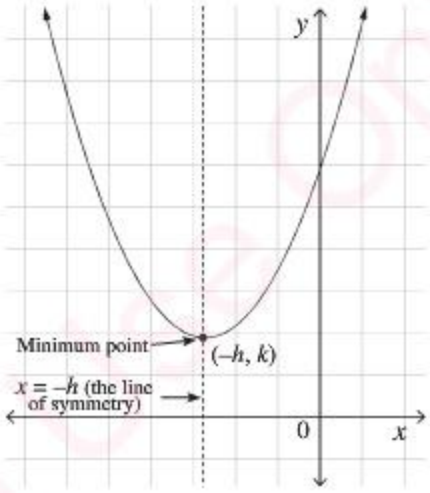

Figure 2.16(a) Graph of $f(x)=a(x+h)^2+k$ where $a>0, h>0$ and $k>0$ showing minimum point and the line of symmetry.

Hence, the graph showing minimum point;

domain of $f(x)=\{\text{all real numbers of } x\}$

and range of $f(x)=\{y: y\geq k\}$

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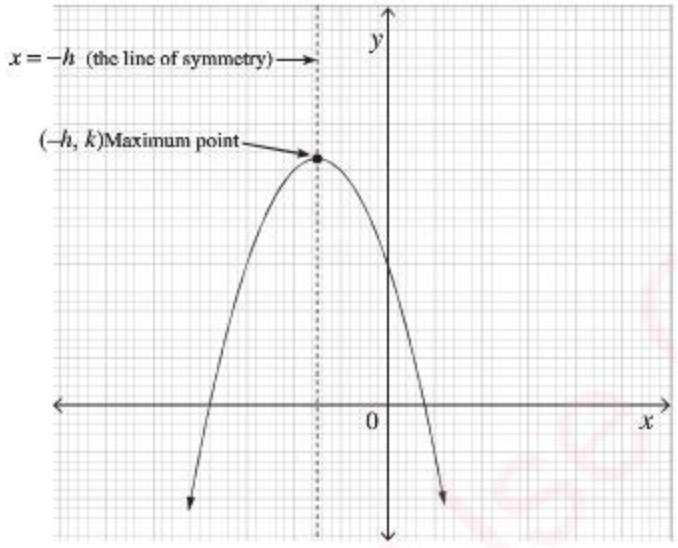

Figure. 2.16(b) Graph of $f(x)=a(x+h)^2+k$, where $a<0, h>0$ and $k>0$ showing maximum point and the line of symmetry.

For the graph showing maximum point:

domain of $f(x)=\{x: x\in\mathbb{R}\}$ and range of $f(x)=\{y: y\le k\}$.

Example 2.14: Given $f(x)=x^2+5x+7$

a) Find the turning point by completing the square

b) Identify whether the turning point obtained in (a) is a maximum or minimum.

c) State the domain and range of the function.

Solution

a) Given $f(x)=x^2+5x+7$, then

$$f(x)=x^2+5x+\left(\frac{5}{2}\right)^2+7-\left(\frac{5}{2}\right)^2$$

$$=x^2+5x+\frac{25}{4}+7-\frac{25}{4}$$

$$=\left(x+\frac{5}{2}\right)^2+7-\frac{25}{4}$$

$$=\left(x+\frac{5}{2}\right)^2+\left(\frac{28-25}{4}\right)$$

$$f(x)=\left(x+\frac{5}{2}\right)^2+\frac{3}{4}$$

Comparing with the form:

$f(x)=a(x+h)^2+k$

then; $a=1,\ h=\frac{5}{2},\ k=\frac{3}{4}$

Therefore, the turning point

$$(-h,k)=\left(-\frac{5}{2},\frac{3}{4}\right)$$

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b) Since the value of a is positive, the turning point is a minimum point.

c) Domain of $f(x)=\{x : x \in \mathbb{R}\}$ and

range of $f(x)=\left\{y : y \geq \frac{3}{4}\right\}$

Example 2.15: Sketch the graph of $f(x)=x^2+2x-8$ using the turning point and the x and $y-$ intercepts.

Solution

Given $f(x)=x^2+2x-8$, then

$$f(x)=x^2+2x+1-8-1$$

$$=(x+1)^2-9$$

The turning point is $(-1,-9)$, and it is a minimum point since $a=1$, which is positive (+).

To obtain x–intercepts, we solve;

$$y=f(x)=0$$

$$0=x^2+2x-8$$

$$0=(x+4)(x-2)$$

$$x=2 \text{ or } x=-4$$

Hence, the curve crosses the x-axis at the point (2, 0) and (-4, 0).

To obtain $y-$ intercept, we find $f(0)$.

Then, $f(0)=0^2+2(0)-8$

$$y=-8$$

Hence, the curve crosses the $y-$ axis at the point (0, –8).

The graph of $f(x)$ is:

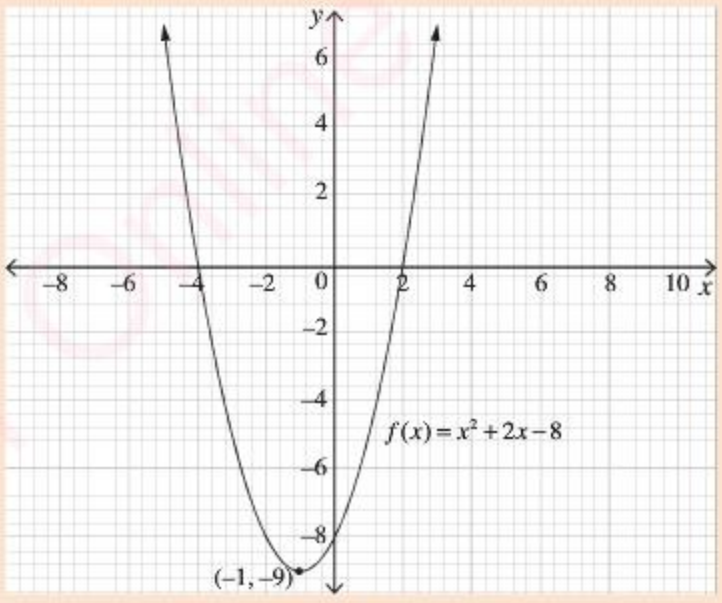

Figure 2.17 Graph of $f(x)=x^2+2x-8$

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Activity 2.3

Consider the quadratic function, $f(x)=ax^2+bx+c$. Identify the following characteristics of the corresponding graph of $f(x)$.

a) Turning point.

b) Domain and range.

c) Direction of opening.

d) The axis of symmetry.

e) The x and $y-$ intercepts.

Exercise 2.6

1. If $f(x)=2(x-3)^2+5$;
   a) Find the turning point of the function.
   b) Identify whether the turning point obtained in (a) is maximum or minimum
   c) Find the line of symmetry.
   d) Find the domain and range of the function.
2. Given $f(x)=x^2-4x+13$:
   a) By completing the square, find and identify the nature of the turning point of $f(x)$.
   b) State the domain and range of $f(x)$.
   c) Sketch the curve of $f(x)$ using the turning point and x and $y-$ intercepts.
3. Draw the graph of the following quadratic functions:
   a) $f(x)=x^2+x-2$
   b) $g(x)=4-6x-3x^2$
4. By completing the square, find the turning point of $f(x)=-2x^2+12x-3$ and state whether it is a maximum or minimum. What is the line of symmetry of this function?
5. A baseball player throws a ball at the height h (in metres) after time t (in seconds). This is approximated by the formula $h=2+28t-5t^2$.
   a) Calculate the time required for the ball to reach a height of 15m.
   b) What is the maximum height reached?

d) Cubic functions

A polynomial function of degree three is called the cubic function. Its general form is $f(x)=ax^3+bx^2+cx+d$, where a, b, c, and d are real numbers and $a\neq 0$. A graph of a cubic function can be drawn by using tabulated values of x and y.

Example 2.16: Draw the graph of $f(x)=x^3-4x$

Solution

Table 2.7 Table of values of $f(x)=x^3-4x$

x	–3	–2	–1	0	1	2	3
$f(x)$	–15	0	3	0	–3	0	15

The graph of function $f(x)=x^3-4x$ is shown in Figure 2.18.

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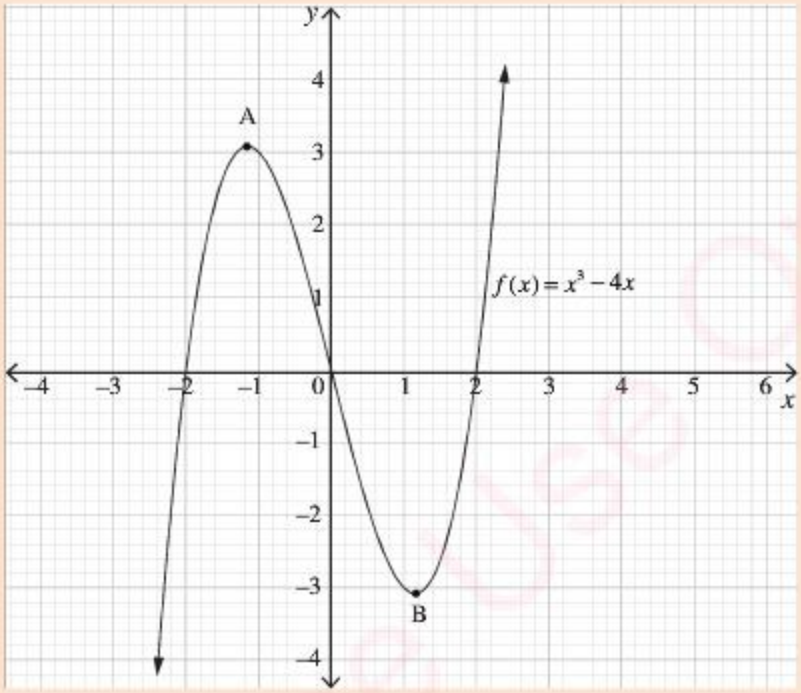

Figure 2.18 The graph of $f(x)=x^3-4x$

Observations from the graph:

a) The turning points A and B are maximum and minimum points respectively

b) The curve crosses the x-axis at three points (-2, 0), (0, 0) and (2, 0).

c) The domain of $f(x)=\{\text{all real numbers}\}$ and the range of $f(x)=\{\text{all real numbers}\}$

e) Step functions

A step function is a function which can be written into other small functions. It is characterized by jumps or steps. Step functions are sometimes called staircase functions.

In plotting the graph of step function, normally, a hollow i.e. (∘) is used to indicate the exclusion of an end point ($<$ or $>$) and filled-in circle or heavy dot i.e. (•) is used to indicate the inclusion of an end point ($\leq$ or $\geq$).

The domain of a step function is the union of all the pieces of functions or the intervals under consideration. The range of a step function is the output of the domain for the considered functions.

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Example 2.17: Consider Table 2.8.

Table 2.8 Cost of mailing parcels

Weight in grams (x)	$0 < x \leq 50$	$50 < x \leq 100$	$100 < x \leq 150$	$150 < x \leq 200$	$200 < x \leq 250$
Cost in Tsh $(f(x))$	500	700	900	1100	1300

Using the above information, draw the graph of step function and determine the domain and range.

Solution

The corresponding graph of $f(x)$ is as follows:

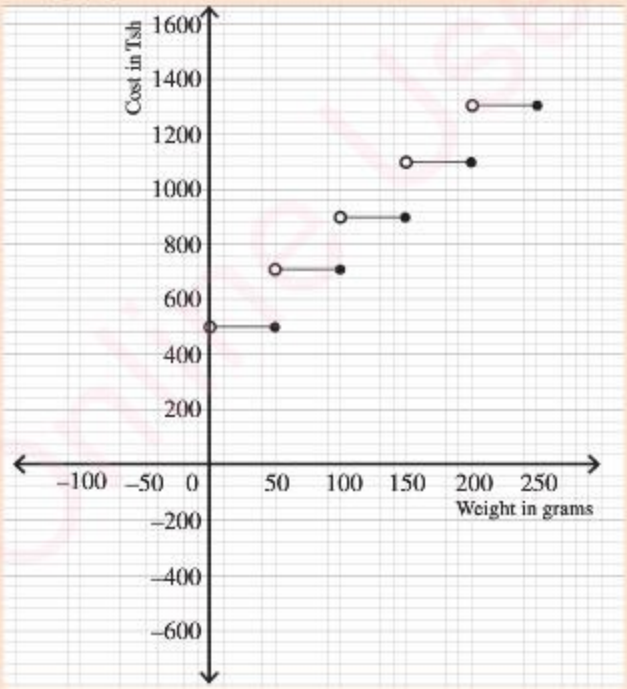

Figure 2.19 Graph of step function indicating the cost of mailing parcels of different weight

Hence, Domain $= \{x : 0 < x \leq 250\}$

Range $= \{500, 700, 900, 1100, 1300\}$

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Example 2.18: Draw the graph of the following step function.

$$f(x)=\begin{cases}-4, & \text{if } x<-2\\ 2, & \text{if } -2\le x<3\\ 5, & \text{if } x\ge 3\end{cases}$$

Hence use the graph to state domain and range of $f(x)$

Solution

The corresponding graph of $f(x)$ is shown in Figure 2.20.

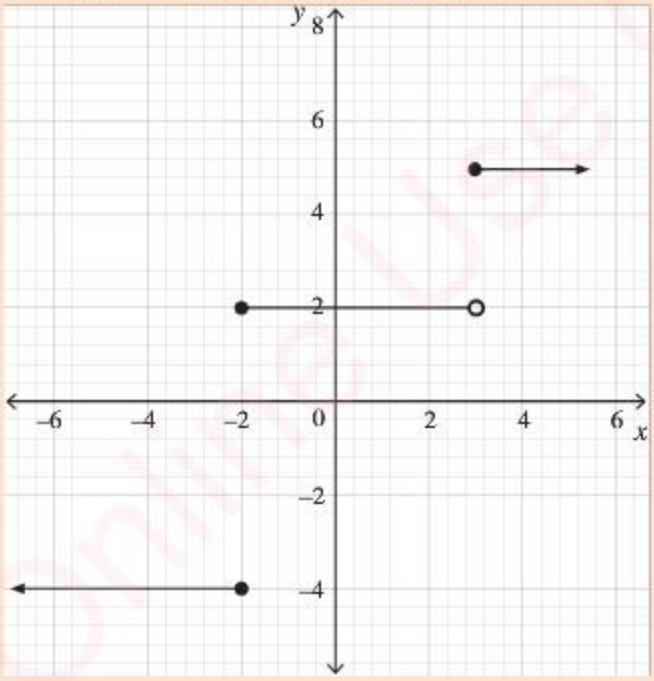

Figure 2.20 The graph of the $f(x)$

The graph shows constant step functions with specified domain in each interval. Domain of $f(x)=\{x:x\in\mathbb{R}\}$ and range of $f(x)=\{-4,2,5\}$.

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Exercise 2.7

1. Draw the graph of $f(x)=x^3-4x^2+3x$.
2. Given $f(x)=(2x-1)\left(x^2-x-2\right)$.
   Draw the graph of $f(x)$, hence state its domain and range.
3. Draw the graph of the following:

$$f(x)=\begin{cases}-2, & \text{if } x>1\\ 1, & \text{if } x=1\\ 3, & \text{if } x<1\end{cases}$$

4. Draw the graph of the function f and hence state its domain and range.

$$f(x)=\begin{cases}2x+1 & \text{if } 0<x<2\\ x^2+1 & \text{if } 2\le x<3\\ x-3 & \text{if } 3\le x<4\end{cases}$$

5. Draw the graph of

$$f(x)=\begin{cases}2, & \text{if } x\le 1\\ 0, & \text{if } -1\le x\le 2\\ 3-x^2, & \text{if } x>2\end{cases}$$

and hence find:

a) The domain and range from the graph of $f(x)$

b) i) $f(-1)$ ii) $f(2)$ iii) $f(3)$

f) *Rational Functions*

It is a function where both numerator and denominator are polynomials.

If $f(x)$ and $g(x)$ are polynomial functions then rational function can be expressed in the form of $\frac{f(x)}{g(x)}$, where $g(x)\neq 0$.

The domain of rational function is the set of all real numbers x except the zeros of the denominator function, i.e. the value of x which make denominator zero. The following are some examples of rational functions:

a) $h(x)=\frac{1}{x+1}$, where $f(x)=1$ and $g(x)=x+1$

b) $h(x)=\frac{1}{x^2+2}$, where $f(x)=1$ and $g(x)=x^2+2$

c) $h(x)=\frac{x}{x^3-1}$, where $f(x)=x$ and $g(x)=x^3-1$

d) $h(x)=\frac{x^2+1}{x^3+2x}$, where $f(x)=x^2+1$ and $g(x)=x^3+2x$

To draw the graph of rational function, consider the following steps:

i) Find the x and y intercepts

ii) Find the vertical and horizontal asymptotes

iii) Test few values in left and right of the vertical asymptotes

Usually graphs of rational functions are characterised by vertical and horizontal asymptotes.

Asymptotes

An asymptote is a line or a curve that approaches a given function infinitesimally closely. That is, as a function approaches infinity, the distance between the line or curve and the function approaches zero.

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Vertical asymptotes

A vertical asymptote of a function $y=f(x)$ is the line $x=a$ at which y approaches $\pm\infty$ as x approaches a from the left and right. It is also known as zeros of the denominator of $f(x)$ (the value of x which makes the denominator of $f(x)=0$). Thus, the graph of $f(x)$ can never touch or cross line $x=a$ (vertical asymptote). A function $y=f(x)$ may have more than one vertical asymptote.

Horizontal asymptotes

The horizontal asymptote of a function $y=f(x)$ is the straight line $y=b$ as y approaches b, x approaches $\pm\infty$. Horizontal asymptotes describe the left and right behaviour of the graph.

The graph of $y=f(x)$ cannot touch or cross the line $y=b$ (horizontal asymptote).

Example 2.19: Draw the graph of the function $f(x)=\frac{1}{x+2}$ and state its domain and range.

Solution

Horizontal Asymptote

$$y=f(x)=\frac{1}{x+2}$$

Divide both numerator and denominator by x to obtain

$$y=\frac{\frac{1}{x}}{\frac{x+2}{x}}=\frac{\frac{1}{x}}{\frac{x}{x}+\frac{2}{x}}$$

$$=\frac{\frac{1}{x}}{1+\frac{2}{x}}$$

as $x\to\infty, \frac{1}{x}\to 0, \frac{2}{x}\to 0$ then $y=\frac{0}{1+0}=0$

$\therefore y=0$ is the horizontal asymptote of $y=\frac{1}{x+2}$

Vertical Asymptote

Since vertical asymptotes are zeros of the denominator of $f(x)$, then;

$x+2=0$

$x=-2$

Hence, vertical asymptote is a straight line $x=-2$.

The corresponding graph of the function $f(x)=\frac{1}{x+2}$ is presented in Figure 2.21.

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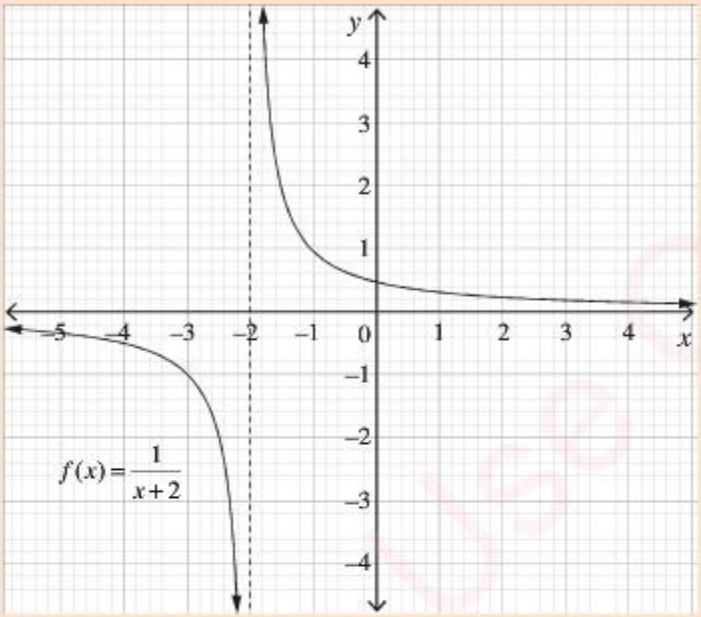

Figure 2.21 Graph of $f(x)=\frac{1}{x+2}$

Domain of $f(x)=\{x\in\mathbb{R}:x\neq-2\}$, Range of $f(x)=\{y\in\mathbb{R}:y\neq0\}$

Example 2.20: Sketch the graph of the function $f(x)=\frac{1}{x}$ and state its domain and range.

Solution

$x-$intercept, when $y=0$, hence

x is undefined

$y-$intercept, when $x=0$, hence

y is undefined

So in both cases the graph will touch neither $x-$axis nor $y-$axis

Vertical asymptote (zeros of the denominator)

From, $f(x)=\frac{1}{x}$, then $x=0$ is the vertical asymptote

Horizontal asymptote

From $f(x)=\frac{1}{x}$, divide both numerator and denominator by x

$$y=\frac{\frac{1}{x}}{\frac{x}{x}}=\frac{\frac{1}{x}}{1}$$

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$$y = \frac{\frac{1}{x}}{1}$$

Since, $x \to \infty, \frac{1}{x} \to 0$ then,

$$y = \frac{0}{1}$$

$$y = 0$$

Therefore, $y = 0$ is the horizontal asymptote of $f(x) = \frac{1}{x}$

Testing few values left and right of $x = 0$

Table 2.9 Table of values of $f(x) = \frac{1}{x}$

x	-2	-1	$-\frac{1}{10}$	0	$\frac{1}{10}$	1	2
$f(x)$	$-\frac{1}{2}$	-1	-10	∞	10	1	$\frac{1}{2}$

as $x \to 0$ from left $f(x) \to -\infty$

$x \to 0$ from right $f(x) \to +\infty$

The graph of $f(x)$ is shown in Figure 2.22.

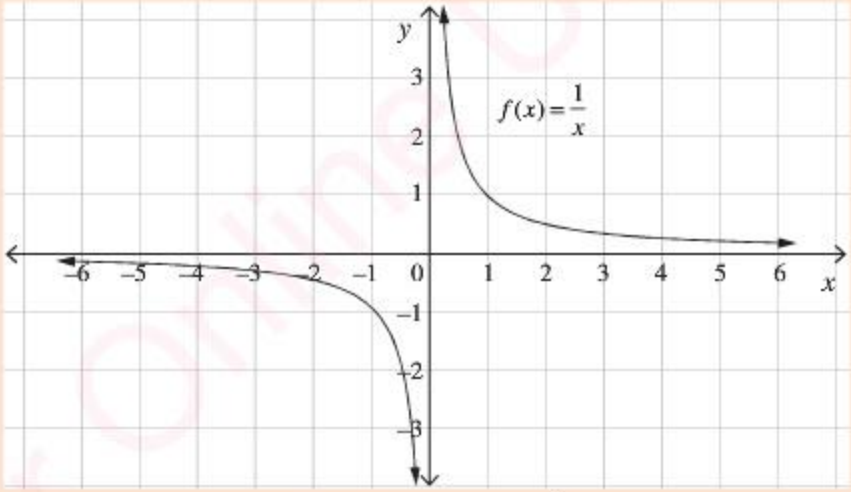

Figure 2.22 Graph of $f(x) = \frac{1}{x}$

Domain of $f(x) = \{x \in \mathbb{R} : x \neq 0\}$ Range of $f(x) = \{y \in \mathbb{R} : y \neq 0\}$

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Example 2.21: Draw the graph of $f(x)=\frac{1}{1-x}$, hence state its domain and range.

Solution

$y-$intercept, when $x=0$, $y=1$, hence $(0,1)$

$x-$intercept, when $y=0$, hence x is undefined which means the graph does not cross $x-$ axis.

Vertical asymptotes

From, $f(x)=\frac{1}{1-x}$, set the denominator $1-x=0$

$$1-x=0$$
$$-x=-1$$
$$x=1$$

$\therefore x=1$, is the vertical asymptote of $f(x)=\frac{1}{1-x}$

Horizontal asymptote

From, $f(x)=\frac{1}{1-x}$, then;

Divide both numerator and denominator by x

$$y=\frac{\frac{1}{x}}{\frac{1-x}{x}}$$

$$y=\frac{\frac{1}{x}}{\frac{1}{x}-1} \text{ as } x\to\infty, \frac{1}{x}\to 0$$

$$y=\frac{0}{0-1}=0,$$

$\therefore y=0$ is the horizontal asymptote of $f(x)=\frac{1}{1-x}$

Testing few values left and right of $x=1$

Table 2.10 Table of values for $f(x)=\frac{1}{1-x}$

x	-1	0	$\frac{99}{100}$	1	$1\frac{1}{100}$	2	3	4
$f(x)$	$\frac{1}{2}$	1	100	∞	-100	-1	$-\frac{1}{2}$	$-\frac{1}{3}$

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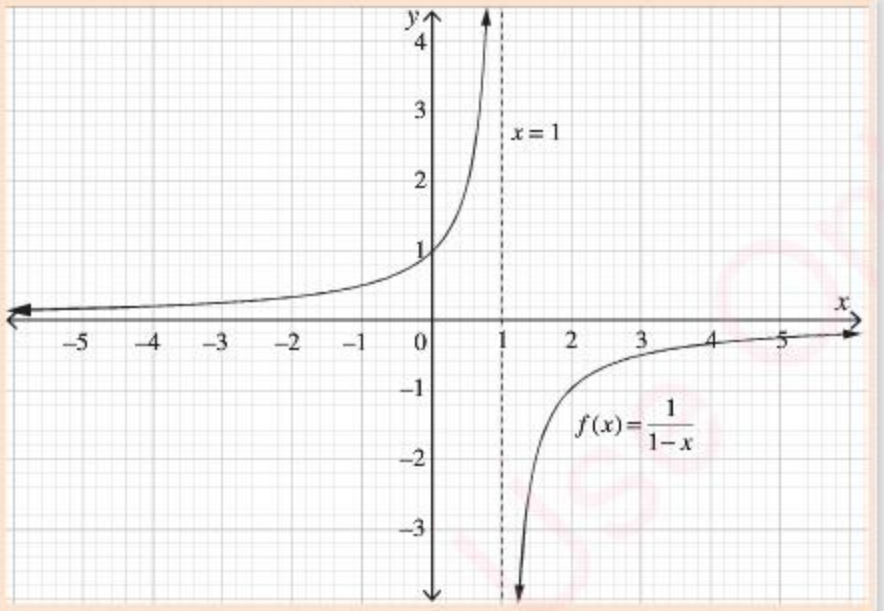

Figure 2.23 Graph of $f(x)=\frac{1}{1-x}$

Domain = $\{x \in \mathbb{R} : x \neq 1\}$ Range = $\{y \in \mathbb{R} : y \neq 0\}$

Exercise 2.8

1. Find the asymptotes and state the domain and range for each of the following:

   a) $f(x)=\frac{1}{x+1}$

   b) $g(x)=\frac{6}{4x-3}$

   c) $g(x)=\frac{2}{1-3x}$

2. Sketch the graph of the following:

   a) $f(x)=\frac{1}{3x}$

   b) $f(x)=\frac{1}{1-2x}$

   c) $g(x)=\frac{3}{2x+8}$

3. A function $f(x)$ is defined as

   $f(x)=\frac{2}{x-3}$ for $x \neq 3$

   a) Why is $x=3$ excluded from the domain of $f(x)$?

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b) Sketch the curve of $y = f(x)$

c) State the domain and range of $f(x)$

4. Given $f(x) = \frac{3}{2-2x}$.

a) For what values of x is $f(x)$ undefined?

b) Describe the behaviour of $f(x)$ as x approaches the values obtained in 4(a)

c) Find x and y intercepts of $f(x)$.

5. Given $f(x) = \frac{1}{ax+b}$, if $f(1) = 4$ and $f(-1) = 2$; find,

a) The values of a and b.

b) The vertical and horizontal asymptotes.

Drawing graphs using a computer

A computer can be used to draw a graph of any type of function. This can be done through the use of various programs or graphing packages. The programs include spreadsheet such as Microsoft excel, Matlab and Maple. These programs differ in capacity and procedures in plotting graphs.

Thus, one needs to study various programs in order to chose the appropriate package for plotting particular types of graphs. In this section, the procedure for drawing graphs using Microsoft excel will be discussed.

Plotting graphs using Microsoft Excel

Sketching the graphs in Microsoft excel has the following advantages and disadvantages.

Advantages

a) It is quick and fast

b) It is accurate

c) It is time saving

d) The plots in Microsoft Excel are neat.

Disadvantages

a) Difficult to troubleshoot or test

b) Susceptible to trivial human errors

In order to plot a graph using the excel sheet, a number of procedures has to be followed. The procedures depend on the type of the function. The procedures are as follows:

1. Defining the coordinates.
2. Typing the data set of independent variables.
3. Creating the formula for values of dependent variables.
4. Plotting the graph.

Graph sketching in Microsoft Excel requires some mathematical symbols to be written with special syntax ; For example x^n is written as $x \wedge n$ and for product, say ax, is $a * x$ where *denote multiplication.

Example 2.22: With a clearly stated procedures, draw the graph of $y = 3x - 1$ using Microsoft Excel.

Solution

The Procedures are:

a) Open Microsoft excel and then type x in cell A1 and y in cell B1

b) Enter x – values

$(\text{e.g}: -4, -3, -2, -1, 0, 1, 2, 3, 4)$ in the cells of column A.

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c) Select the x – values and click the "Formula tab", "Create from Selection", and finally click "Top row" to automatically generate the name of the selected cells.

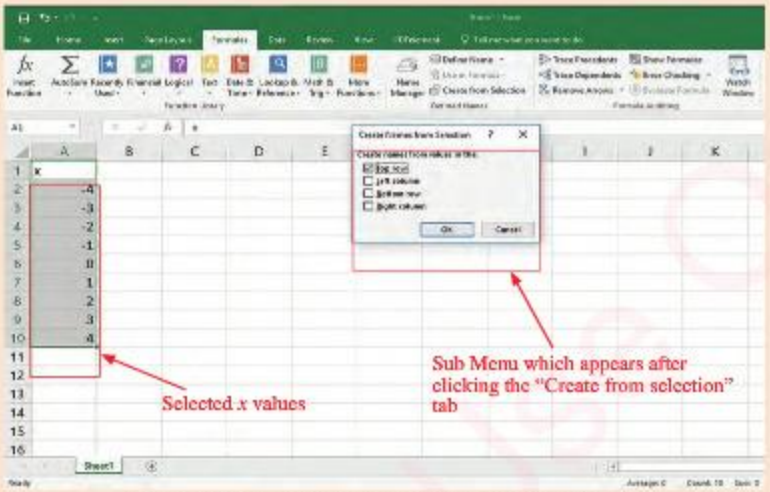

Figure 2.24 Microsoft Excel sheet displaying steps (a) - (c)

d) Set the formula = (3*A2–1) at B2 for calculating the value of y, then drag it to apply the formula to other cells of column B. That is dragging the formula from B2 to B10.

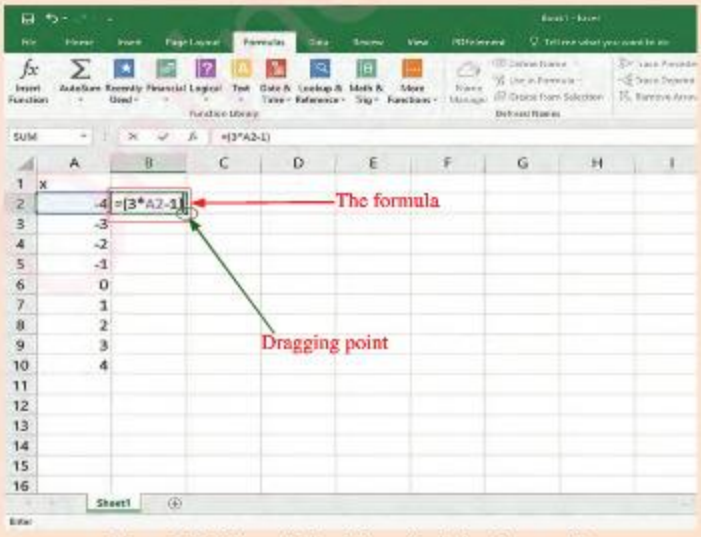

Figure 2.25 Microsoft Excel sheet displaying the step (d)

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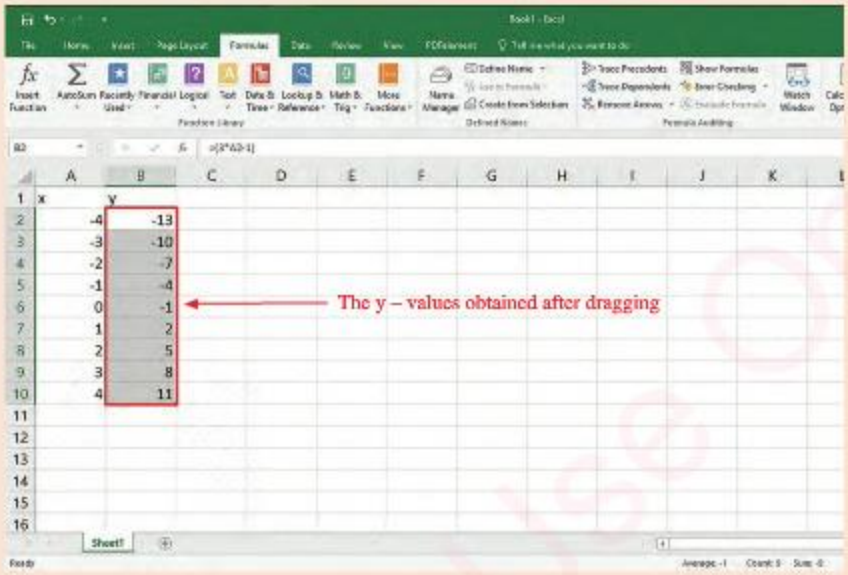

Figure 2.26 Microsoft Excel sheet displaying the results after dragging the formula

e) Select the x and y values as illustrated in the Figure 2.27 then click the Insert Tab. On the insert Menu, place the pointer on the icons and click the icon which displays the instruction "Scatter with smooth Lines and Markers"

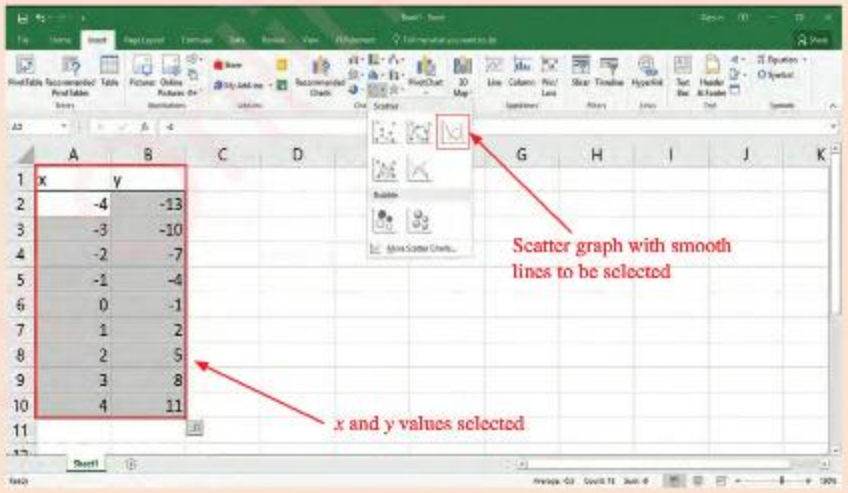

Figure 2.27 Microsoft Excel sheet displaying the step (e) - (f).

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f) After clicking the option "scatter with smooth lines and markers", the graph of $y=3x-1$ will be displayed as shown in the Figure 2.28.

Figure 2.28 Microsoft Excel sheet displaying the graph of $f(x)=3x-1$ after step (f).

The axes and grid lines of Figure 2.28 can be formatted and the line graph can be named. To format a particular axis, right click on any number of that axis then select "Format Axis" option and do the formatting of your choice such as line styles or line colours. To format grid lines, right click any number on the particular line and then select "Add Minor Grid lines" or "Add Major grid lines". To format and name the graph, right click any point on the graph and select "Format Data Series" and then do the formatting of your choice. Figure 2.29 is a result of some formatting on axes and grid lines on Figure 2.28.

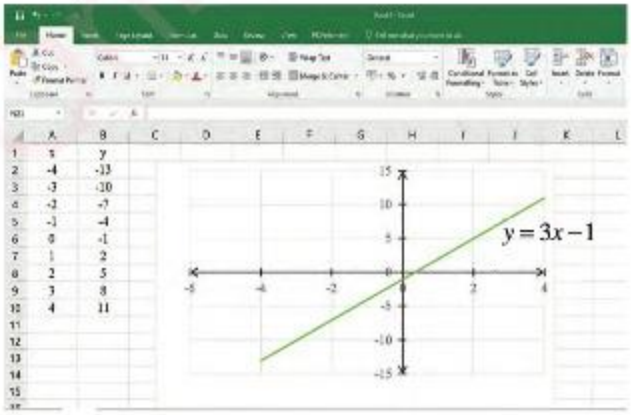

Figure. 2.29 Formatted axes and grid lines of the graph of $f(x)=3x-1$

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Exercise 2.9

1. Draw the graph of each of the following functions using Microsoft Excel computer package.
   a) $f(x)=-2x^2+2x-6$
   b) $g(x)=x^3-4x^2+x$
2. Draw the graph of each of the following functions using spreadsheet and compare with the hand drawn graph.
   a) $f(x)=-3$
   b) $g(x)=5-7x$
   c) $h(x)=\dfrac{1}{4x+5}$

2.3 Importance of functions

Functions is one of the topics with many applications in real life. The following are some areas where functions can be applied:

a) In laboratories: During measurement of temperature using a thermometer, the experimenter has to wait for a while before taking reading. This implies that temperature depends on time of heat supply.

Thus, the working principle of a thermometer can be defined by a function. Actually, the length of the thread on the scale of the thermometer varies linearly with time until the maximum point is reached. Thus, an experimenter with the knowledge of function is expected to use the thermometer properly.

b) In motion: Velocity (speed) and displacement (distance) of a moving body are determined by using functions of time. The velocity V of a moving body with a constant acceleration a is given by $V(t)=u+at$ and its distance S is given by $S(t)=ut+\frac{1}{2}at^2$. These are linear and quadratic functions , respectively.

c) In supply and demand: Any quantity of a product that a producer is willing to supply to the market corresponds to various prices during some periods of time. The higher the price per unit product, the larger the quantity. Suppose p is the price per unit product and q is the quantity supplied, this relation can be presented mathematically as $p=a+bq$ where a and b are the initial prices and the coefficient of the quantity supplied, respectively.

d) In hospitals: The trend of child growth is recorded on a clinic card where the weight of the child in kilograms is located against the age. The points are joined by lines to give a graph. Therefore, a person with the knowledge of functions has advantage in both recording and interpreting the clinic cards than the ones without knowledge of functions.

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Revision exercise 2

1. Draw the graph of the polynomial $P(x)=2x^3-3x^2-12x+13$
2. Find all factors of $f(x)=x^3-8$ if $D(x)=x-2$ divides $f(x)$
3. What is the degree of the following polynomials?
   a) $f(x)=5x-1-4x^2+3x^3$ b) $g(t)=t^3+4t^2+5$ c) $h(y)=y^2+5y+1$
   d) $h(x)=x^4+7x^2+1$ e) $g(x)=x^2+3x^3+x+7$
4. Identify each of the following expressions whether they are polynomials or not? Explain why?
   a) $\sqrt{x}+2x$ d) $x^{\frac{3}{2}}+4x-x^3$
   b) x^4+5x^3+2x+1
   c) $\dfrac{x^2-9}{x-3}$
5. A rational function is defined by $f(x)=\dfrac{1}{3x-2}$. Find,
   a) the asymptotes
   b) the intercepts
   c) the domain and range without sketching
6. Given that $x\in\{0,1,3,5\}$, find the corresponding values of y in each of the following:
   a) $y=\sqrt{9-x}$ b) $y=2x^2+3$ c) $y=\dfrac{1}{x^2+1}$
7. If the domain defined by the rule $f:x\to x^2+1$ is $\{3,4,5,6\}$, find its range.
8. Given $f(x)=\sqrt{3-x}$
   a) Find the domain of $f(x)$ in the form $x<b$ where b is a real number.
   b) Find $f(-2)$, $f(-1)$, and $f(1)$ leaving surd in your answer.
9. Draw the graph of $f(x)=x(1-x)$ and then state its domain and range.
10. The function f is defined as:

$$f(x)=\begin{cases}2x+12 & \text{if } -2<x\leq 1\\ x^2 & \text{if } 1<x\leq 2\\ 10-x & \text{if } 2<x<3\end{cases}$$

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a) Sketch the graph of $f(x)$ and determine its domain and range

b) Find $f(1)$, $f(2)$ and $f\left(2\frac{1}{2}\right)$

11. A function f is defined by $f(x)=\frac{1}{1-x}$ where $x\neq 1$;

a) Why is $x=1$ excluded from the domain of $f(x)$?

b) Sketch the graph $y=f(x)$.

12. Show that the equation $x^3+x^2-7=0$ may be arranged in the form of

$x=\sqrt{\frac{a}{x+b}}$. What are the values of a and b?

13. Suppose $A=\{-2, -1, 0, 1, 2\}$ is the domain of the function $f:x\rightarrow\frac{2}{5}x$. Represent this function pictorially.

14. Sketch the graph of $y=f(x)$ where $f(x)=1-x^2$ when $0\leq x\leq 1$, and use the sketch to find the domain and range of $f(x)$.

15. Given the following functions:

a) $g:x\rightarrow\frac{1}{x-3}$ b) $f:x\rightarrow\frac{3}{4-x}$ c) $h:x\rightarrow\frac{2}{1+x}$

Find: (i) Asymptotes (ii) Domain and range (iii) x and $y-$intercepts and draw the graph of each function given.

16. The function f is defined as $f(x)=\begin{cases}x & \text{if } 0\leq x\leq 1\\ x(x-2) & \text{if } 1<x<3\end{cases}$

Sketch the graph of $f(x)$ and hence find the domain and range.

17. Sketch the graph of $f(x)=\begin{cases}x-1 & \text{if } x<0\\ 0 & \text{if } x=0\\ x^2-1 & \text{if } x>0\end{cases}$ and find:

a) the domain and range of $f(x)$

b) Find (i) $f(-1)$ (ii) $f(0)$ (iii) $f(3)$

18. Sketch the graph of $f(x)=\begin{cases}x^2-1 & \text{if } x>0\\ 0 & \text{if } x=0\\ x^2-1 & \text{if } x<0\end{cases}$ and then find:

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a) the domain and range

b) (i) $f(0)$ (ii) $f(-3)$ (iii) $f(11)$

19. Given $f(x)=x^2-3x+5$. Without drawing the graph, find:
    a) the domain and range of the function
    b) the turning points and state whether it is maximum or minimum.

20. If the function $f(x)=x^3-2x^2-5x+6$ is divided by $x-1$ find all the factors of $f(x)=x^3-2x^2-5x+6$.

21. Given $f(x)=-2x^2-4x+11$:
    a) Find the turning point and state whether it is a maximum or minimum point.
    b) Find the values of x and y–intercepts.
    c) Sketch the graph of the function using the turning point and the intercepts.

22. a) Given $f(x)=2x+3$, $g(x)=x-5$ and $h(x)=4x+1$, find
        (i) $f(x)+g(x)+h(x)$
        (ii) $f(x)g(x)h(x)$

    b) Divide the polynomial $P(x)=3x^2-7x+4$ by $x+1$.

    c) Given $f(x)=\dfrac{3}{x-2}$ and $g(x)=\dfrac{2}{x+1}$, find $f(x)-g(x)$.

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Chapter Three

Algebra

Introduction

Algebra deals with the representation of numbers and quantities in formulae and equations by using symbols and letters. It has a number of applications in real-life situations such as determining interests in financial institutions, estimating profit and loss in business, scale drawing, and currency conversions, among other applications. In this topic, you will learn about simultaneous equations, finite and infinite series of Arithmetic Progression (AP) and Geometric Progression (GP), ratios, variations and applications of Algebra in solving daily life problems.

3.1 Simultaneous equations

The term simultaneous equations refer to a set of two or more equations with two or more variables whose values satisfy both equations in a set. However, the discussion shall base not only on a system of two linear equations with two unknowns, but also on a system of two unknowns containing one linear equation and one quadratic equation.

Formulating and solving linear simultaneous equations

Word problems can be solved by expressing them in the form of equations in relation to number of unknowns. The methods used in solving linear simultaneous equations are substitution method, elimination method, graphical method and by using matrix method or Cramer's rule. This section deals with two methods namely elimination and substitution. However, students are requested to use other methods/ techniques for solving algebraic equations.

a) Elimination method

This method involves manipulation of the two equations so that when combined, either the x variable or the y variable is eliminated. The resulting equation with only one unknown can then be solved.

Thereafter, it is easy to solve one of the equations so that when it is combined with the other equation, either the value of x or y will be obtained. In the following example, the variable x will be eliminated giving a solution for y.

Example 3.1: Think of the two integers such that twice the first minus the second gives five and their sum is one. Formulate the linear equations and find the two integers?

Solution

Let x and y be the two integers, respectively. The two linear equations are deduced as:

$$\begin{cases} 2x - y = 5 \\ x + y = 1 \end{cases} \quad (3.1)$$

From equation (3.1),

$$2x - y = 5 \quad \text{and} \quad (3.2)$$

$$x + y = 1 \quad (3.3)$$

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Multiply equation (3.2) by 1 and equation (3.3) by 2 so as to make the coefficients of x in (3.2) and (3.3) equal. That is;

$$2\left|\begin{array}{l}2x-y=5\\x+y=1\end{array}\right.$$

$$=\begin{array}{ll}2x-y=5 & (3.4)\\2x+2y=2 & (3.5)\end{array}$$

To eliminate x, subtract equation (3.5) from equation (3.4). That is,

$$-\left|\begin{array}{l}2x-y=5\\2x+2y=2\end{array}\right.$$

$$-3y=3$$

The resulting equation will be:

$$-3y=3$$

$$\therefore y=-1$$

Again recall equation (3.2) and (3.3), the coefficients of y are equal.
To eliminate y from equation (3.2) and (3.3), add the two equations.

$$+\left|\begin{array}{l}2x-y=5\\x+y=1\end{array}\right.$$

$$3x=6$$

So the resulting equation will be:

$$3x=6$$

$$\frac{3x}{3}=\frac{6}{3}$$

$$x=2$$

Since $x=2$ and $y=-1$, therefore, the two integers are 2 and –1.

b) Substitution method

The method involves substitution where in one of the equations, one variable is expressed in terms of the other and substituted into the other equation.

This gives an equation with just one unknown, which can be easily solved. The value obtained is substituted in one of the original equations leading to an equation with one unknown.

Recall Example 3.1: $\begin{cases}2x-y=5\\x+y=1\end{cases}$

From equation (3.2), making y the subject of the equation yields;

$$y=2x-5 \qquad (3.6)$$

Substitute the value of y in equation (3.3) to obtain;

$$x+(2x-5)=1$$

$$3x-5=1$$

$$3x=6$$

$$\therefore x=2$$

Then, substitute $x=2$ in equation (3.6)

$$y=2\times(2)-5$$

$$y=4-5$$

$$\therefore y=-1$$

Since $x=2$ and $y=-1$, therefore, the two integers are 2 and –1.

Solving simultaneous equations with linear and quadratic equations

Word problems may involve solving simultaneous equations where one is linear equation and the other is a quadratic equation.

Example 3.2: There are two numbers such that their sum is 10 and the sum of their squares is 52. Formulate this as simultaneous equations and find the value of the two numbers.

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Solution

Let x and y be the two numbers, then the two equations will be

$$\begin{cases} x+y=10 \\ x^2+y^2=52 \end{cases} \quad (3.7)$$

The equations of the system (3.7) can be written as follows:

$$x+y=10 \quad (3.8)$$

$$x^2+y^2=52 \quad (3.9)$$

The convenient method for solving the equations (3.8) and (3.9) is substitution method.

Making y the subject in equation (3.8), and then substitute it in equation (3.9) yields

$$y=10-x \quad (3.10)$$

$$x^2+(10-x)^2=52$$

By expanding, we have,

$$x^2+100-20x+x^2=52$$

Resulting into $2x^2-20x+48=0$, then dividing by 2 gives

$$x^2-10x+24=0 \quad (3.11)$$

The result is a quadratic equation. Solving equation (3.11) by factorization through splitting the middle term:

Now, $x^2-6x-4x+24=0$

and, $x(x-6)-4(x-6)=0$

then, $(x-4)(x-6)=0$

Either $(x-4)=0$ or $(x-6)=0$

$x=4$ or $x=6$

Substitute the values of x into equation (3.10).

Now, if $x=4$, then $y=10-4$

and if $x=6$, then $y=10-6$

$x=6$ then $y=10-6=4$

and if $x=4$ then $y=10-4=6$

Therefore, the two numbers are 4 and 6.

Example 3.3: The sum of two whole numbers is 5 and the difference of their squares is 5. Find the two numbers.

Solution

Let the two whole numbers be x and y, respectively such that $x>y$. Then;

$$\begin{cases} x+y=5 \\ x^2-y^2=5 \end{cases} \quad (3.12)$$

The equations of the systems (3.12) can be written as follows:

$$x+y=5 \quad (3.13)$$

$$x^2-y^2=5 \quad (3.14)$$

In equation (3.13), make x the subject

$$x=5-y \quad (3.15)$$

Substitute equation (3.15) into equation (3.14).

$$(5-y)^2-y^2=5$$

Thus,

$$25-10y+y^2-y^2=5$$

$$25-10y=5$$

Then,

$$10y=20$$

$$\therefore\ y=2$$

Substitute the value of y in equation (3.15). That is, $x=5-y$.

then, $x=5-2$

$x=3$

Hence, $x=3$ and $y=2$

Therefore, the whole numbers are 3, and 2.

Alternative method;

Recall; $x+y=5$

$x^2-y^2=5$

Using equation (3.14), applying difference of two squares, we have:

$$(x-y)(x+y)=5 \quad (3.16)$$

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Substitute equation (3.13) into equation (3.16),

$$(x-y)5=5$$

$$x-y=1 \qquad (3.17)$$

Then, solve linear simultaneous equations (3.13) and (3.17) by elimination method:

$$+\begin{cases} x+y=5 \\ x-y=1 \end{cases}$$

Then $2x=6$

$x=3$

Substituting $x=3$ into either equation (3.13) or (3.17).

That is, $3+y=5$

$\therefore y=2$

Therefore, $x=3$ and $y=2$

Hence, the two whole numbers are 2 and 3.

Example 3.4: The sum of two numbers is twelve and the product of the same numbers is twenty, what is the difference between the two numbers?

Solution

Let the two numbers be

a and b such that $a>b$

Equations are as follows:

$$\begin{cases} a+b=12 \\ ab=20 \end{cases} \qquad (3.18)$$

The equations of the system (3.18) can be written as follows:

$$a+b=12 \qquad (3.19)$$

$$ab=20 \qquad (3.20)$$

From equation (3.19), make a the subject

$$a=12-b \qquad (3.21)$$

Substitute a in equation (3.15). That is,

$$(12-b)\times b=20$$

Thus, $12b^2-b^2=20$ or $b^2-12b+20=0$

Now, by factorization method, we have;

$$b^2-10b-2b+20=0$$

$$(b-10)(b-2)=0$$

$b=10$ or $b=2$

Therefore, the value of $b=2$ or $b=10$

Substitute these values of b in equation (3.21)

when $b=10$, then $a=12-10=2$

when $b=2$, then $a=12-2=10$

Since $a>b$, therefore, the value of $a=10$ and $b=2$

Hence, the difference of a and b is:

$$a-b=10-2=8$$

Therefore, the difference between the two numbers is 8.

Exercise 3.1

Solve the following simultaneous equations:

1. $\begin{cases} x+y=5 \\ x-y=1 \end{cases}$
2. $\begin{cases} x+y=7 \\ 2x+3y=4 \end{cases}$
3. $\begin{cases} 2x+3y=14 \\ 3x+2y=16 \end{cases}$
4. $\begin{cases} x-y=6 \\ 2x-3y=13 \end{cases}$
5. $\begin{cases} x-y=3 \\ x^2-y^2=5 \end{cases}$
6. $\begin{cases} x+y=15 \\ x^2-y^2=75 \end{cases}$
7. $\begin{cases} x+y=7 \\ xy=5 \end{cases}$
8. $\begin{cases} 3x+2y=27 \\ x^2-4xy+4y^2=9 \end{cases}$
9. The difference between two numbers is 10 and the difference between their squares is 40. Formulate a system of simultaneous linear equations and find the numbers.

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10. A kindergarten school placed two orders with the stationery dealers. The first order was for 26 boxes of books and 8 boxes of coloured pencils with the total amount of Tsh 974,000. The second order was 12 boxes of books and 4 boxes of coloured pencils with the total amount of Tsh 464,000. Formulate a system of simultaneous equations and calculate the cost of a box of books and a box of coloured pencils.

3.2 Finite and infinite series

The terms sequence and series are sometimes used interchangeably although they have different meaning. A sequence is an ordered set of numbers that follow a certain rule to determine the subsequent terms while a series is a summation of the terms of a sequence. There are two types of series namely; finite and infinite series. A finite series is a summation of a limited number of terms while an infinite series has unlimited number of terms.

Example 3.5: State, with reasons, whether the following are:

a) Sequence or series

b) Finite or infinite

(i) 4, 9, 16, …

(ii) $1+2+3+4+5+\ldots+9$

(iii) 1, 3, 5, 7, 9 ,…, 21, 23

(iv) $2+4+6+8+\ldots$

Solution

a) i) It is a sequence because it is arranged as a pattern with commas.

ii) It is a series because it is written as a sum of terms.

iii) It is a sequence because it is arranged as a pattern of numbers with commas.

iv) It is a series because it is written as a sum of terms.

b) i) It is an infinite sequence because it has unlimited number of terms (has no end).

ii) It is a finite series because it has a limited number of terms (has an end).

iii) It is a finite sequence since it has a limited number of terms.

iv) It is an infinite series since it has no end.

General term

The general terms is a common rule which describes the terms in a sequence or series. It is usually expressed in terms of its position.

For example; in the sequence 1, 2, 3, 4,…

the 1st term is 1, the 2nd term is 2, the 3rd term is 3 …

Thus, at any position r, the r^{th} term is r.

Hence the general term of this sequence is r, i.e $T_r = r$ where $r = 1, 2, 3, 4, \ldots$

Similarly, other examples of general terms are;

a) The general term for the sequence of even numbers; $2, 4, 6, 8, \ldots$ is $T_r = 2r$

b) The general term for the sequence of odd numbers; $1, 3, 5, 7, \ldots$ is $T_r = 2r - 1$

c) The general term for the sequence of squares; $1, 4, 9, 16, \ldots$ is $T_r = r^2$.

Sigma notation

Sigma notation is a Greek letter which represents the summation of terms for a series of numbers. It is denoted by the symbol $\sum$.

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To express a series in sigma form, the general term is used with the sigma notation indicating the positions where the sum of the terms begins and ends.

i.e Series $= \sum_{r_0}^{n} T_r$

where r_0 is the initial value of r and n is the last value of r.

Examples:

1. $1+2+3+4+5=\sum_{1}^{5} r$

2. $6+8+10+...+20=\sum_{1}^{8}(2r+4)$

Remark: At position $r = n$, the general term is called the n^{th} term.

Example: The n^{th} term of the sequence 1, 3, 5, 7, ... 2n – 1, ... is $T_n = 2n-1$.

The n^{th} term is obtained by replacing r with n to the general term.

For example; 1 + 2 + 3 + 4 +...+ 100 can be written in sigma notation as $\sum_{a=1}^{100} a$, which means that the sum of all terms from $a = 1$ to $a = 100$.

Properties of sigma notation

a) $\sum_{1}^{n} c = nc$ b) $\sum_{1}^{n} ar = a\sum_{1}^{n} r$

c) $\sum_{1}^{n}\left(ar \pm br^2\right) = a\sum_{1}^{n} r \pm b\sum_{1}^{n} r^2$

where a, b and c are constants.

Example 3.6: Express the following series by using sigma notation.

a) $1^2+2^2+3^2...+20^2$

b) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...$

Solution

a) $1^2+2^2+3^2...+20^2 = \sum_{m=1}^{20} m^2$

b) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...=\sum_{n=1}^{n}\frac{1}{2^n}$

Example 3.7: Write the following in an expanded form.

a) $\sum_{j=1}^{n} x_j$ b). $\sum_{k=1}^{\infty} k^3$

Solution

a) $\sum_{j=1}^{n} x_j = x_1+x_2+x_3+...+x_n$

b) $\sum_{k=1}^{\infty} k^3 = 1^3+2^3+3^3+...$

Exercise 3.2

1. Identify which of the following are sequences or series (finite or infinite) and state the reason.
   a) 2, 4, 6, 8, ... 24
   b) 2, 3, 5, 8, ... 27
   c) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...$
   d) 1+2+6+24+...50040
2. Write the following series in expanded form:

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a) $\sum_{k=1}^{5} k^2$ b) $\sum_{m=2}^{4} m^3$ c) $\sum_{x=1}^{n} (1+x)^2$

3. Express the following series in sigma notation.
   a) $2+4+6+...+2n$
   b) $1-2+4-8+16-32$
   c) $1+\frac{2}{3}+\frac{3}{9}+\frac{4}{27}+\frac{5}{81}$

4. Show that $\sum_{r=1}^{5} (r^2+4r-1) = 110$

5. Evaluate $\sum_{r=1}^{5} r^3$

The nth term of an Arithmetic Progression (AP)

An Arithmetic Progression (AP) is a series or a progression where each new term after the first is obtained by adding a constant (called the common difference) to the preceding term. If the first term of the arithmetic progression is a_1 then its sequence is $a_1,\ a_1+d,\ a_1+2d,\ a_1+3d,...$ hence the n^{th} term is $a_n = a_1+(n-1)d$. Therefore, the finite series of AP is:

$$a_1+(a_1+d)+(a_1+2d)+(a_1+3d)+...+(a_1+(n-1)d)$$

Example 3.8: Find the common difference for each of the following:

a) $2+5+8+...+32$

b) $4+6+8+10+...+82$

Solution

Using the general formula for the n^{th} term:

$$a_n = a_1+(n-1)d$$

a) where

$a_1 = 2,\ a_2 = 5,\ n = 2,\ d = ?$

Then $5 = 2+d$

$d = 5-2$

$\therefore d = 3$

Alternatively;

$d = a_2 - a_1 = a_3 - a_2 = 5-2 = 8-5 = 3$

b) where

$a_1 = 4,\ a_2 = 6,\ n = 2,\ d = ?$

Then $6 = 4+d$

$d = 6-4$

$d = 2$

Alternatively;

$d = a_2 - a_1$

$= a_3 - a_2$

$= 6-4$

$= 8-6$

$= 2$

Activity 3.1

Fill the numbers (1-100) in Table 3.1 in ascending order from left to right.

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Table 3.1 Numbers in ascending order

a) Describe the pattern of the numbers along the rows.

b) Describe the pattern of the numbers along the columns.

The Sum of an Arithmetic Progression

Let a be the first term of an AP, a_n be the n^{th} term, and S_n be the sum of the first $n-$terms and n be the number of terms.

From the general form of AP,

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_n - d) + a_n \tag{3.22}$$

By reversing the order,

$$S_n = a_n + (a_1 - d) + (a_n - 2d) + \ldots + (a_1 + 2d) + (a_1 + d) + a_1 \tag{3.23}$$

Then, adding equation (3.22) and (3.23)

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n) \tag{3.24}$$

Since there are n–terms on the right hand side of the equation (3.24), then:

$$2S_n = n(a_1 + a_n)$$

$$S_n = \frac{n(a_1 + a_n)}{2}$$

Since a_n is the n^{th} term of AP, then $a_n = a_1 + (n-1)d$.

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Hence, $S_n = \frac{n\left[a_1 + a_1 + (n-1)d\right]}{2}$

$$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

$$\therefore S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] \text{ or } S_n = \frac{n(a_1 + a_n)}{2} \quad (3.25)$$

Example 3.9: The fifth term of an arithmetic progression is six times the second term and the sum of the second and third terms is eleven. Find:

a) the first term
b) the common difference
c) the sum of the first ten terms

Solution

a) Given that $a_5 = 6a_2$ then,

$$a_5 + 4d = 6(a_1 + d)$$

$$a_1 + 4d = 6a_1 + 6d$$

$$-5a_1 - 2d = 0$$

$$\therefore 5a_1 + 2d = 0 \quad (3.26)$$

Also;

$a_2 + a_3 = 11$ then,

$$(a_1 + d) + (a_1 + 2d) = 11$$

$$2a_1 + 3d = 11 \quad (3.27)$$

Equation (3.26) and (3.27) can be solved as follows:

$$\begin{cases} 5a + 2d = 0 \\ 2a_1 + 3d = 11 \end{cases}$$

Multiply equation (3.26) by 3 and equation (3.27) by 2,

$$15a_1 + 6d = 0 \quad (3.28)$$

$$4a_1 + 6d = 22 \quad (3.29)$$

Subtract equation (3.29) from (3.28)

$$11a_1 = -22$$

$$\therefore a_1 = -2$$

b) Using equation (3.26), substitute $a_1 = -2$

Now $5(-2) + 2d = 0$

$$-10 + 2d = 0$$

then, $2d = 10$

$$\therefore d = 5$$

c) The sum of the first ten terms

Consider, $S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right]$

Then, $S_{10} = \frac{10}{2}(2(-2) + (10-1)5)$

$$S_{10} = 5(-4 + 45)$$

$$S_{10} = 5 \times 41$$

$$\therefore S_{10} = 205$$

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Exercise 3.3

1. Find the number of terms in the following series
   a) 3–2–7...–42
   b) 1+3+5+...+99
   c) $1\frac{1}{8}+1\frac{1}{4}+1\frac{3}{8}+...+3\frac{1}{2}$
2. Find the sum of the following Arithmetic progressions
   a) 1+3+5+...+101
   b) 2+7+12+...+77
   c) –10 –7 – 4 –...+50
3. The second term of an AP is 15 and the fifth term is 21. Find the common difference, the first term and the sum of the first ten terms.
4. The fourth term of an AP is 18 and common difference is -5. Find the first term and sum of the first sixteen terms.
5. Find the sum of the even numbers, divisible by three, lying between 400 and 500.
6. The third and fifth terms of an AP are 8 and 14, respectively. Find the sum of the first eleven terms.
7. How many terms of the AP $10+8+6+...$ will give a total of -60. Find the sum of the first ten terms.

The n^{th} term of a Geometric Progression (GP)

A geometric progression is a series or progression where each new term after the first is obtained by multiplying the preceding term by a constant r, called the common ratio. If the first term of the sequence is G_1, then the terms of a GP are G_1, G_1r, G_1r^2, G_1r^3,... Hence the n^{th} term is $G_n = G_1r^{n-1}$.

The sum of finite series for Geometric Progression

Let G_1 be the first term of a GP, r be the common ratio, n be number of terms and S_n be sum of the first n terms of the GP. The n^{th} term of a GP is $G_n = G_1r^{n-1}$. Thus,

$$S_n = G_1 + G_1r + G_1r^2 + G_1r^3 + ... + G_1r^{n-1} \quad (3.30)$$

Multiplying by r both sides of the series in equation 3.30 results into;

$$rS_n = rG_1 + G_1r^2 + G_1r^3 + G_1r^4 ... + G_1r^{n-1} + G_1r^n \quad (3.31)$$

Subtracting equation (3.31) from (3.30) result into:

$$S_n - rS_n = G_1 - G_1r^{n-1} \quad (3.32)$$

Equation (3.32) may be written as:

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$$(1-r)S_n = G_n(1-r^n) \tag{3.33}$$

Dividing equation (3.33) by (1–r) both sides:

$$S_n = G_1\left(\frac{1-r^n}{1-r}\right), \text{appropriate when } r<1. \tag{3.34}$$

Or subtracting (3.30) from (3.31) and simplify, result into;

$$S_n = G_1\left(\frac{r^n-1}{r-1}\right), \text{appropriate when } r>1. \tag{3.34}$$

Example 3.10: The sum of certain number of terms of GP having first term 1 and common ratio 2 is 255. How many terms are there?

Solution

Given that,

$G_1 = 1,\ r = 2$ and $S_n = 255,\ n = ?$

Now, $S_n = G_1\left(\frac{r^n-1}{r-1}\right)$

where $r>1$

Then, $255 = 1\left(\frac{2^n-1}{2-1}\right) = 2^n - 1$

$2^n - 1 = 255$

$2^n = 255 + 1$

$2^n = 256$

$2^n = 2^8$

Equating the exponents, gives $n = 8$

Therefore, there are 8 terms.

Infinite geometric series

Consider the following infinite geometric progression:

a) $1+4+16+64+256+\ldots$

b) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$

For the first series, $r = 4$, the terms becomes larger and larger as the number of terms "n" increases, and therefore its sum will increase rapidly.

For the second series, $r = \frac{1}{2}$, the terms of the series become smaller and smaller as the number of terms "n" increases. Its sum will be approaching a certain definite number since the higher terms will be very small (approaching zero).

It is then possible to find the sum to infinity (S_∞) when the value of r lies between -1 and 1 (i.e. when r is a fraction, such that $|r|<1$.

Recall the formula, $S_n = \frac{G_1(1-r^n)}{1-r}$

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If r is a fraction, the value of r^n approaches 0 as n approaches infinity. Symbolically we can write $r^n \to 0$ as $n \to \infty$.

So that the sum to infinity of a GP (S_∞) is given by:

$$S_\infty = \frac{G_1(1-0)}{1-r}$$

$$S_\infty = \frac{G_1}{1-r}, \text{ where } |r| < 1$$

Example 3.11: Convert the following recurring decimals to series and find fractions in lowest terms

a) $0.\dot{0}5$ b) $0.\dot{2}\dot{3}$ c) $8.0\dot{3}\dot{7}$

Solution

a) $0.\dot{0}5$ means 0.055555...
which can be written as:
$\frac{5}{100}+\frac{5}{1000}+\frac{5}{10000}+...$

This is a GP with $G_1 = \frac{5}{100}$ and $r = \frac{1}{10}$, then:

$$S_\infty = \frac{\left(\frac{5}{100}\right)}{\left(1-\frac{1}{10}\right)}$$

$$S_\infty = \frac{\left(\frac{5}{100}\right)}{\left(\frac{9}{10}\right)} = \frac{5}{100}\times\frac{10}{9} = \frac{1}{18}$$

$$\therefore S_\infty = \frac{1}{18}$$

Hence, $0.0\dot{5} = \frac{1}{18}$

b) $0.\dot{2}\dot{3}$ means 0.2323...,
and can be written as:
$\frac{23}{100}+\frac{23}{10000}+\frac{23}{1000000}+...$

This is a GP with $G_1 = \frac{23}{100}$, $r = \frac{1}{100}$

The sum to infinity is thus given by:

$$S_\infty = \frac{G_1}{1-r}$$

$$\text{then } S_\infty = \frac{\left(\frac{23}{100}\right)}{\left(1-\frac{1}{100}\right)}$$

$$S_\infty = \frac{23}{100}\times\frac{100}{99}$$

$$\therefore S_\infty = \frac{23}{99}$$

Hence, $0.\dot{2}\dot{3} = \frac{23}{99}$.

c) $8.0\dot{3}\dot{7}$ means 8.0373737...

This can be written as

$$8+\left(\frac{37}{1000}+\frac{37}{100000}+\frac{37}{10000000}+...\right)$$

This is a GP with,

$$G_1 = \frac{37}{1000} \text{ and } r = \frac{1}{100}$$

From $S_\infty = \frac{G_1}{1-r}$ where $|r| < 1$,

$$= G_1\left(\frac{1}{1-r}\right)$$

Then, $\frac{37}{1000}\left(\frac{1}{1-\frac{1}{100}}\right) = \frac{37}{1000}\times\frac{100}{99}$

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$$=\frac{37}{990}$$

Adding this to the whole value 8 gives

$$8\frac{37}{990}$$

Converting $8\frac{37}{990}$ to improper fraction gives $\frac{7957}{990}$

Hence $8.0\dot{3}7=\frac{7957}{990}$

b)

Using $S_n=\frac{G_1\left(1-r^n\right)}{1-r}$ where $|r|<1$ then,

$$S_5=\frac{2\left(1-\left(\frac{1}{2}\right)^5\right)}{1-\frac{1}{2}}=\frac{2\left(1-\frac{1}{32}\right)}{\frac{1}{2}}$$

$$=4\left(\frac{32-1}{32}\right)=\frac{4\times31}{32}$$

$$=\frac{31}{8}=3\frac{7}{8}$$

$$\therefore S_5=3.875$$

Example 3.12: The sum to infinity of a GP is 4 and the common ratio is $\frac{1}{2}$.

Find: a) The first term

b) The sum of the first five terms.

Solution: Given $S_\infty=4,\quad r=\frac{1}{2}$

a) Using $S_\infty=\frac{G_1}{1-r}$ where $|r|<1$

then, $4=\frac{G_1}{1-\frac{1}{2}}$

$$4=\frac{G_1}{\frac{1}{2}}$$

$$G_1=4\left(\frac{1}{2}\right)$$

$$\therefore G_1=2$$

Remark: Whole numbers are added after finding the sum to infinite of the series resulting from the fractional part of the recurring decimal number.

Compound interest

Compound interest is the method of paying interest where the interest for each period is added to the principal before interest is calculated for the next period. With this method, the principal grows as the interest is added to it.

This method is used in investments such as savings account and bonds. An understanding of compound interest is important not only for people planning careers with financial institutions, but also for anyone planning to invest money.

The formula for computing the amount accumulated when interest is compounded can be developed as follows.

Let P be the principal borrowed at a compound interest rate of $r\%$ per annum for a period of t years.

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The amount A_1 (principal + interest), at the end of one year will be

$$A_1 = P + Pr = P(1+r)$$

At the end of second year, and subsequent years, the amount are given by

$$A_2 = A_1 + A_1 r = A_1(1+r) = P(1+r)(1+r) = P(1+r)^2$$

$$A_3 = A_2 + A_2 r = A_2(1+r) = P(1+r)^2(1+r) = P(1+r)^3$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$A_t = A_{t-1} + A_{t-1}r = A_{t-1}(1+r) = P(1+r)^t$$

Thus, the amount A that accumulated on principal P after t years at $r\%$ rate of compound interest per annum will be

$$A_t = P(1+r)^t \tag{3.35}$$

Remark:

1. If the interest is compounded annually, then $A_t = P(1+r)^t$
2. If the interest is compounded half-yearly, then $A_t = P\left(1+\frac{r}{2}\right)^{2t}$
3. If the interest is compounded quarterly, then $A_t = P\left(1+\frac{r}{4}\right)^{4t}$

In general, if the interest is compounded n times a year, then $A = P\left(1+\frac{r}{n}\right)^{nt}$ where n represent the number of times the interest is compounded per annum.

Example 3.13: A person deposited Tsh 480000 in a bank for 3 years at 4% per annum. Find the compound interest when the interest is payable

a) annually b) half-yearly c) quarterly

Solution

$P = 480,000$, $r = \frac{4}{100} = 0.04$, $t = 3$.

a) Since the interest is payable yearly then

$$A = P(1+r)^t$$
$$= 480,000(1+0.04)^3$$
$$= 539,934.72$$

Now, since $I = A - P$

then, $= 539,934.72 - 480,000$

$=$ Tsh 59,934.72

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Therefore, the compound interest paid annually is Tsh 59,934.72.

b) Since the interest is payable half-yearly

$$A = P\left(1+\frac{r}{2}\right)^{2t}$$

$$=480{,}000\left(1+\frac{0.04}{2}\right)^{2\times 3}$$

$$=540{,}557.96$$

Since $I = A - P$

then $=540{,}557.96 - 480{,}000$

$=$ Tsh 60,557.96

Therefore, the compound interest paid half-yearly is $=$ Tsh 60,557.96.

c) Since the interest is payable quarterly,

$$A = P\left(1+\frac{r}{4}\right)^{4t}$$

$$=480{,}000\left(1+\frac{0.04}{4}\right)^{4\times 3}$$

$$=540{,}876.01$$

Since, $I = A - P$

then, $=540{,}876.01 - 480{,}000$

$=$ Tsh 60 557.96

Therefore, the compound interest paid quarterly is Tsh 60,557.96.

Example 3.14: What percentage rate of compound interest per annum will Tsh 600,000 amount to Tsh 800,000 in 5 years if the interest is calculated half yearly.

Solution: Given that

$A = 800{,}000$, $P = 600{,}000$, $t = 5$ and $n = 2$

From $A = P\left(1+\frac{r}{n}\right)^{nt}$

$$800{,}000 = 600{,}000\left(1+\frac{r}{200}\right)^{2\times 5}$$

$$\frac{800{,}000}{600{,}000} = \left(1+\frac{r}{200}\right)^{10}$$

$$\left(\frac{8}{6}\right)^{\frac{1}{10}} = \left(1+\frac{r}{200}\right)$$

$$\left(1+\frac{r}{200}\right) = 1.0292$$

$$\frac{r}{200} = 1.0292 - 1$$

$$\therefore\ r = (0.0292)(200) = 5.8$$

Hence, the rate of interest is 5.8% per annum.

Exercise 3.4

1. For each of the following series, determine whether it is an AP or a GP. Find the sum of the first ten terms.
   a) $1+2+4+\ldots$
   b) $2+4+6+\ldots$
2. The third and sixth terms of a GP are 10 and 80, respectively. What is the sum of the first six terms?
3. Write down the sum of the first n-terms of the following series, and deduce the sums to infinity
   a) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$
   b) $\frac{13}{100}+\frac{13}{1000}+\frac{13}{10000}+\ldots$
   c) $0.5+0.05+0.005+\ldots$
   d) $0.43+0.0043+0.000043+\ldots$
   e) $64-16+4-\ldots$

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4. Convert the following repeating decimals into fractions in lowest terms:

   a) $0.\dot{2}$ b) $0.\dot{2}\dot{1}$ c) $5.\dot{2}$

   d) $4.4\dot{9}$ e) $3.5\dot{7}\dot{3}$

5. a) If the sum to infinity of a GP is three times the first term, what is the common ratio?

   b) The second term of a GP is 24 and its sum to infinity is 100. Find the two possible values of the common ratio and the corresponding first terms.

6. Show that if $\log a$, $\log b$, $\log c$ are consecutive terms of an AP, then a, b, c are in GP.

7. Write the following series in expanded form:

   a) $\sum_{k=1}^{4} x$ b) $\sum_{r=1}^{5} 2r-1$ c) $\sum_{k=1}^{100} 2k$

   d) $\sum_{j=1}^{n} y_j$ e) $\sum_{k=1}^{\infty} (-1)^k k$

   f) $\sum_{m=1}^{\infty} \frac{m}{m+1}$ g) $\sum_{k=1}^{n} ar^{k-1}$

   h) $\sum_{m=0}^{7} \frac{m+1}{2m+1}$ i) $\sum_{k=2}^{\infty} (-1)^k \frac{k}{k-1}$

8. Express the following in sigma notation:

   a) $2+4+6+8+\ldots$

   b) $1+3+5+7$

   c) $1^2+2^2+3^2+4^2+\ldots+10^2$

   d) $1+2+3+4+\ldots+200$

   e) $2+\frac{3}{2}+\frac{4}{3}+\ldots+\frac{n+1}{n}$

   f) $k+k^2+k^3+k^4+k^5$

   g) $1\times2+2\times3+3\times4+\ldots+n(n+1)$

   h) $-1+4-9+16-25+\ldots$

9. In how many years will a sum of money treble itself at 4% per annum, interest compounded annually?

10. Prove that in order the sum of money to double itself in 10 years by investment at compound interest payable annually the interest should be approximately 7% per annum.

3.3 Ratios and variations

Ratios are used to compare two or more quantities of the same units. It can be presented in a form of division of two numbers, fraction or as a percentage.

Example 3.15: At a certain secondary school, there are 1,100 girls and 700 boys. Find the ratio of:

a) Girls to boys

b) Boys to girls.

c) Explain the meaning of the answer obtained in (a).

Solution

Let G represent number of Girls and B represent number of Boys.

a) Ratio of G to B

$$G = \frac{n(G)}{n(B)} = \frac{1100}{700} = \frac{11}{7}$$

Ratio of G to $B = 11:7$

b) Ratio of B to $G = \frac{n(B)}{n(G)} = \frac{700}{1100} = \frac{7}{11}$

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Ratio of B to G=7 : 11

c) The meaning of the answer obtained in (a) is that in every group of eighteen students there are eleven girls and seven boys.

Application of ratios

Ratios have numerous applications in real life. Some of these applications are currency conversion and scale drawings. For instance, ratios are used in business to compare profit and loss; in sports ratio can be used to compare wins, losses and draws.

a) Currency conversion

In order to do transactions, the knowledge of currency conversion is needed. This can be achieved by using the concept of ratio. Table 3.2 portrays different currency rates of other countries against the Tanzanian shillings.

Table 3.2 Foreign currency rates against the Tanzanian shilling as of August 07, 2017

Foreign country	Foreign currency	Tanzania rates (Tsh)
Kenya	1 Ksh	21.55
Uganda	1 Ush	0.62
South Africa	1 Rand	167.39
America	1 U.S dollar	2238.80
Europe	1 Euro	2642.80
United kingdom	1 Pound	2922.12
Japan	1 Yen	20.20
China	1 Yuan	333.27
India	1 Rupee	35.12
Saud Arabia	1 Riyal	597.09

Example 3.16: If ten U.S dollars are exchanged for Tsh 22,388:

a) Find the rate of exchange.

b) How many Tanzanian shillings one can get from 25,000 U.S dollars?

c) How many U.S dollars are needed to exchange with Tsh 770,000?

Solution

a) Rate of exchange = Ratio of U.S dollars to Tanzania shillings

$$\text{Rate of exchange} = \frac{\$10}{\text{Tsh } 22{,}388}$$

Rate of exchange \$1 : Tsh 2,238.8

This means that one U.S dollar is exchanged for Tsh 2,238.80.

b) If \$1 : Tsh 2,238.8

\$2,500 : x

It follows that:

$$x = \frac{\text{Tsh } 2{,}238.80 \times \$25{,}000}{\$1}$$

x = Tsh 55,970,000

∴ Tsh 55,970,000 will be received from the exchange of 25,000 U.S dollars.

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c) If \$1 : Tsh 2,238.8

y : Tsh 770,000

It follows that;

$$y = \frac{\text{Tsh } 770{,}000 \times \$1}{\text{Tsh } 2{,}238.8}$$

$$y = \$343.93$$

$\therefore$ 343.93 US dollars received for exchange of Tsh 770,000.

b) Scale

A scale of a map is usually written as a ratio of the distance on a piece of a paper (map) to actual distance on the ground. For example, if the scale of a map is 1:50 000, it means that 1 unit (cm) on a piece of paper represents 50 000 units (cm) on the actual ground.

Example 3.17: The scale of a map is given by 1:10,000. How long in metres will the actual ground distance be for a map distance of 2.5 cm? What length on a map will represent 0.75 km on the ground?

Solution

Given Scale 1:10 000

This means that 1cm on a map represents 10,000 cm on the ground.

So 2.5 cm will represent x cm on the ground. That is,

1 cm : 10,000 cm

2.5 cm : x

It follows that,

$$x = \frac{2.5 \times 10{,}000 \text{ cm}}{1}$$

$$x = 25{,}000 \text{ cm}$$

But 1 m = 100 cm

Therefore, 2.5 cm on the map represents 250 cm on the actual ground.

But 1 km = 100,000 cm

then, 0.75 km = 0.75 × 100,000 cm

= 75,000 cm

Then,

1 cm = 10,000 cm

y : 75,000 cm

$$y = \left(\frac{75{,}000}{10{,}000}\right) \text{ cm}$$

$$\therefore y = 7.5 \text{ cm}$$

Hence, a distance of = 0.75 km will be represented by 7.5 cm on the map.

Example 3.18: A crown of gold and silver is made in the ratio of $1\frac{1}{2} : 2\frac{1}{4}$ by mass.

If the mass of gold is 24 g, find:

a) The total mass of the crown

b) The mass of silver in the crown.

Solution

a) The ratio is $1\frac{1}{2} : 2\frac{1}{4}$, given that mass of gold is 24 g.

Therefore, sum of ratios $= \frac{3}{2} + \frac{9}{4}$

$$= \frac{15}{4}$$

Let c be the total mass of crown,

Thus, $$24 = \left(\frac{3}{2} \times \frac{4}{15}\right)c$$

$$c = \frac{5 \times 24}{2}$$

$$c = 60 \text{ g}$$

$\therefore$ The total mass of crown is 60 g.

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b) The mass of silver = The total mass of crown – The mass of gold

$= (60 - 24)\text{ g}$

$= 36\text{ g}$

The mass of silver in the crown $= 36$ g

Alternatively;

$$\text{Mass of silver} = \left(\frac{9}{4} \div \frac{15}{4}\right) \times 60\text{ g}$$

$$= \left(\frac{9}{4} \times \frac{4}{15}\right) \times 60\text{ g}$$

$$= \frac{9}{15} \times 60\text{ g}$$

$$= 3 \times 12\text{ g}$$

$$= 36\text{ g}$$

Therefore, the mass of silver in the crown is 36 g.

Variations

Variations deals with explaining how quantities in mathematical language change with respect to one or more quantities. There are two primary types of variations; direct variation and inverse variation. However, there is an extension of similar concepts; joint variation and combined variation.

a) Direct variation

The concept of direct variation is determined by direct relationship of the quantities that the increase of one quantity leads to the increase of the other and the vice versa. Two quantities, say A and B are said to have direct proportion relationship if the ratio is always a constant number and that when A increases causes B to increase and the vice versa.

That is; $\frac{A}{B} = k$, where k is constant.

Also, $\frac{A_1}{B_1} = \frac{A_2}{B_2} = k$

Example 3.19: Two tractors can cultivate 20 acres of land per day. If three tractors of the same make are added, how many acres of land will be cultivated?

Solution

Consider the table below

Tractor (A)	Amount of Land (B)
2	20
5	x

Using the above illustrations:

$$\frac{A_1}{B_1} = \frac{A_2}{B_2} = k$$

$$\frac{2}{20} = \frac{5}{x}$$

$x = 50$ acres

$\therefore$ 50 acres will be cultivated.

Hence, this shows that when the number of tractors is increased, the number of acres will also increase (direct variation).

In general, if y varies directly as x, it is written as $y \alpha x$ where α represent proportionality sign.

Then, introduce the constant of proportionality k from the definition of direct proportional.

Thus, $\frac{y}{x} = k$ or $y = kx$

This is the equation of direct variation.

Linear direct variation, $y = kx$ where $k = 2$, its graph will be as shown in Figure 3.1.

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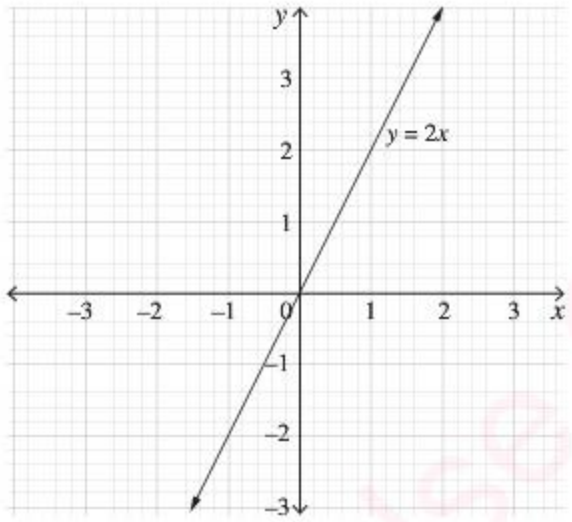

Figure 3.1 The graph showing direct variation

Example 3.20: If y varies directly as x, and $x=8$, $y=2$. Find:

a) y when $x=1$

b) x when $y=20$

Solution

Given, $y \propto x$ then $\frac{y}{x}=k$

So, $y=kx$

Substituting $x=8$ and $y=2$ gives

$$k=\frac{y}{x}=\frac{2}{8}=\frac{1}{4}$$

The equation becomes: $y=\frac{1}{4}x$

a) Substituting $x=1$ yields $y=\frac{1}{4}\times 1=\frac{1}{4}$

b) Substituting $y=20$ in the equation above, $20=\frac{1}{4}x$

$$x=20\times 4$$
$$\therefore x=80$$

b) Inverse variation

Two quantities, say A and B, are said to have an inverse proportion relationship if their product is always a constant. This means that as A increases, B decreases and the vice versa. Therefore, when two variables or quantities change in opposite directions, that is inverse variation.

That is $A_1 \times B_1 = A_2 \times B_2 = ... = A_n \times B_n$

Generally, if y varies inversely as x, that can be written as:

$$y \propto \frac{1}{x}$$

It is read as y is inversely proportional to x.

Eliminating the proportionality sign $\propto$, the standard equation of inverse variation is:

$$y=\frac{k}{x} \quad \text{or} \quad xy=k$$

The sketch of the graph showing inverse variation for $k=2$ is shown below in Figure 3.2.

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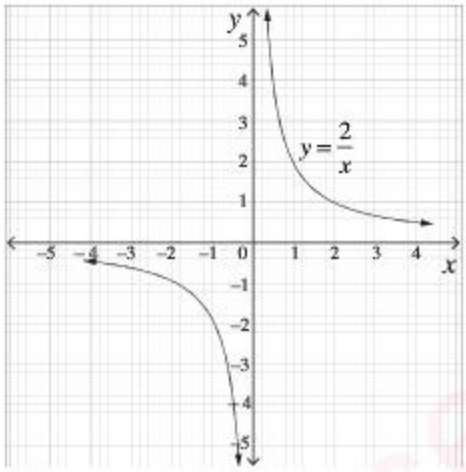

Figure 3.2 The graph showing inverse variation

Example 3.21: If y varies inversely as x and $x=10$ when $y=50$, find:

a) y when $x=\frac{1}{5}$

b) x when $y=10$

Solution: The equation is:

$$y\alpha\frac{1}{x} \text{ or,}$$

$$y=\frac{k}{x} \text{ or,}$$

$$xy=k$$

$$y\alpha\frac{1}{x}$$

$$k=10\times50$$

$$\therefore k=500$$

a) From the given equation,

$$y=\frac{500}{x}$$

But $x=\frac{1}{5}$

thus, $y=\frac{500}{\left(\frac{1}{5}\right)}=500\times5$

$$\therefore y=2{,}500$$

b) From $y=\frac{500}{x}$

$$10=\frac{500}{x} \text{ or } 10x=500$$

$$\therefore y=50$$

c) *Joint and combined variation*

Joint variation is just an extension of direct variation because sometimes one quantity can depend on more than one quantity. This is known as joint variation. If "y varies jointly with x and z" it is an example of joint variation. This can be written as

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$y = kxz$. However, there are also combined variations which involve both direct and inverse variation. For example, if "y varies directly with x and inversely with z," then it is a combined variation.

From $y \alpha \frac{x}{z}$, it can be written as follows:

$y = k\frac{x}{z}$ where k *is* constant

Example of joint variation: A volume of cylinder is directly proportional to the square of radius and height i.e. $V = kr^2h$.

Example of combined variation: Pressure is directly proportional to force and inversely proportional to area i.e. $P = k\frac{F}{A}$.

Example 3.22: Suppose z varies directly as x and inversely as the square root of y. Given $z = 2$, $x = 8$ and $y = 4$.

Find:

a) z when $x = 1$ and $y = \frac{1}{4}$

b) y when $z = 3$ and $x = 2$

Solution: From the question: $z \alpha \frac{x}{\sqrt{y}}$

$$z = \frac{kx}{\sqrt{y}} \text{ and,}$$

$$k = \frac{z\sqrt{y}}{x}$$

$$k = \frac{2 \times \sqrt{4}}{8}$$

$$= \frac{4}{8}$$

$$= \frac{1}{2}$$

Using the equation: $z = \frac{x}{2\sqrt{y}}$

a) $z = \frac{1}{2 \times \sqrt{\frac{1}{4}}} = \frac{1}{2 \times \frac{1}{2}} = \frac{1}{1} = 1$

b) $3 = \frac{2}{2 \times \sqrt{y}} = \frac{1}{\sqrt{y}}$

$$9 = \frac{1}{y}$$

$$\therefore y = \frac{1}{9}$$

Exercise 3.5

1. Divide 300,000 shillings to three people in the ratio 2:3:5.
2. If $x : 20 = 27 : 54$, find the value of x
3. The ratio of boys to girls in a certain school is 2:5. If there are 1400 boys, what is the total number of students in the school?
4. One Dubai Dhiram is equivalent to Tsh 4000 while one Euro is equivalent to Tsh 2000. How many Euros will be obtained for the exchange of 1,000,000 Dubai Dhiram?
5. The scale of a map is 2cm to 8m. Write this without units in a ratio form in its lowest terms.
6. A certain juice is made from the mixture of ten grams of fruit flavour with a litre of water. How many litres of water are needed for 1kg of fruit flavours?
7. Suppose y varies directly as the square of x, and $y = 98$ when $x = 7$. Find x when $y = 32$

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8. Suppose y varies inversely as x, and $y=4$ when $x=8$. Find y when $x=\frac{1}{2}$.
9. A varies directly as B and inversely as C; and $A=5$, $B=10$, and $C=1$. Find A when $B=20$ and C=10.
10. Given that Q is direct proportional to t and inversely proportional to $\sqrt{s}$ and $Q=7$ when $t=2$ and $s=4$. Find Q when $t=4$ and $s=9$.

Revision exercise 3

1. Solve the following simultaneous equations.
   a) $\begin{cases} 2x-7y=5 \\ x^2+y^2=5 \end{cases}$ b) $\begin{cases} x+y=5 \\ xy=15 \end{cases}$
2. Find the sum of the following series.
   a) $5+15+45+...+3645$ b) $8+4+0+...+-48$
   c) $\frac{1}{2}+\frac{1}{6}+\frac{1}{18}+...+\frac{1}{1458}$ d) $3+7+11+...+47$
3. Find the sum to infinity of the following.
   a) $1+\frac{1}{3}+\frac{1}{9}+...$ b) $1-\frac{1}{2}+\frac{1}{4}+...$ c) $27+18+12+...$
4. What is the value of k if three numbers $k+3$, $2k+6$ and 8 are consecutive terms of an AP?
5. $P+3$, $P+8$, and $P+18$ are the 3^{rd}, 4^{th}, and 5^{th} terms of a GP, respectively. Find the value of P and the common ratio.
6. The ratio of quantities A and B is 2:5. If A increases by 40% and B increases by 30%. Find the new ratio.
7. Six people working 8 hours a day takes ten days to finish a certain job. How long will it take for two people working 12 hours per day to finish the same job?
8. The resistance R of a conductor varies directly as the length L and inversely as the cross-sectional area A of a conductor. Given that: $R=10\Omega$, when $L=2\,\text{m}$ and $A=0.005\,\text{m}^2$. Find R when $L=20\,\text{m}$ and $A=20\,\text{cm}^2$.
9. If you deposit Tsh 5000 into an account paying 6% annual interest compounded monthly, how long will it take until there is Tsh 8000 in the account?
10. Find the ratio of x to that of y in the equation $2x^2+7xy+3y^2=0$.

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11. Consider two numbers such that the second number minus one the result is three times the first number. If seven minus the square of the first number equals the second number subtracting twice the first number;
    a) formulate a system of equations for the possible numbers.
    b) find the two unknown numbers.
12. Write the following series using sigma notation:
    a) $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^n}$
    b) $1^2+3^2+5^2+7^2$
    c) $f_1x_1+f_2x_2+f_3x_3+...+f_nx_n$
13. Y varies directly as the square of x and inversely as the cube root of z. if z is 24 when $z=8$ and $x=4$, find y when $z=27$ and $x=5$.
14. Four children Shilla, Makala, Mkumbo, and Eva were given pocket money by their mother in the ratio 2:3:1:4, respectively. If Makala received Tsh 60,000, how much did Shilla, Mkumbo, and Eva receive?
15. Solve for x in each of the following proportions:
    a) $15:x=10:7$
    b) $5-3x:-8=3:2x$
16. If y varies directly as x and that $y=6$ when $x=2$, express y as a function of x.
17. If the ratio of m to n is $\frac{2}{3}$, find the ratio of $5m+3n$ to $7m-4n$.

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Chapter
Four

Differentiation

Introduction

Differentiation is a process of determining the rate at which one quantity changes with respect to the change in another quantity. In real-life situation, differentiation may be applied in solving problems concerning population modelling, fluid flow, dynamical systems, electricity, economics, and business. In this chapter, you will learn about the meaning of derivatives (also known as differentiation), use of techniques of differentiation in finding derivatives, use of differentiation in identifying critical points of curves, and application of differentiation in solving real life problems.

4.1 Differentiation by first principle

Differentiation by the first principle is the process of determining the derivative (slope) of the function by computing the limit of the ratio of small changes in dependent variable y to small changes in independent variable x as small increment denoted by h goes close to zero. Figure 4.1 illustrates the idea of derivative of a function $y = f(x)$.

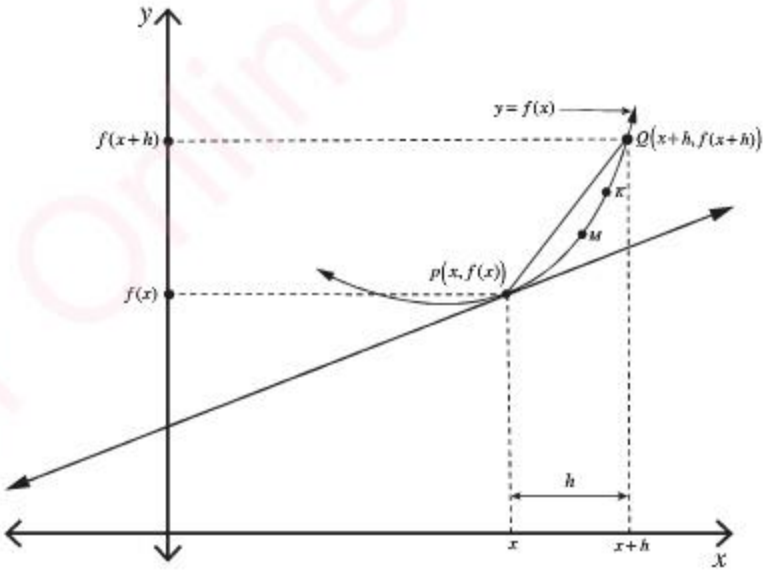

Figure 4.1 Illustration of the gradient of a curve $y = f(x)$

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Let $P(x, f(x))$ and $Q(x+h, f(x+h))$ be two points on the curve.

Gradient of line $\overline{PQ} = \dfrac{f(x+h)-f(x)}{x+h-x}$

$= \dfrac{f(x+h)-f(x)}{h}$

Now, if Q is moved nearer to P, say to points K, M ..., then the chord $\overline{PK}$, $\overline{PM}$, ... will be coming close and close to the tangent line drawn to touch the curve at point P. The action of moving point Q towards point P decreases the value of the increment h. Thus, as h approaches zero, (i.e. $h \to 0$), the line $\overline{PQ}$ approaches the tangent line at P. Hence, the gradient of chord $\overline{PQ}$ approaches the gradient of the tangent at P.

Gradient of tangent at point P can then be said to be the limiting value of the gradient of the chord $\overline{PQ}$ as $h \to 0$. This can be written as:

$$= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

However, the gradient of the curve at point P is equal to the gradient of the tangent at the same point P.

Therefore, the gradient of the curve is written as $= \lim_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$

The gradient of the curve is called the gradient function or derivative function or derived function because it is derived from the original function.

The process of finding the derivative of a function is called **differentiation**. The symbol for derivative of a function is

$$f'(x) = \frac{df}{dx} \text{ or } y'(x) = \frac{dy}{dx}.$$

Now, using the symbols above, first principles of the derivative of $f(x)$ is given as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \text{ or}$$

$\dfrac{dy}{dx} = \lim_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$, which means a function is differentiated with respect to x.

Example 4.1: Use the first principle of differentiation to find the derivative of $f(x) = x^2$.

Solution

By using the first principle of differentiation:

i.e. $f'(x) = \lim_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \dfrac{(x+h)^2 - x^2}{h}$

$= \lim_{h \to 0} \dfrac{x^2 + 2xh + h^2 - x^2}{h}$

$= \lim_{h \to 0} \dfrac{h(2x+h)}{h}$

$= \lim_{h \to 0} (2x+h)$

$= 2x$

Therefore, $f'(x) = 2x$ is the derivative of $f(x) = x^2$.

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Example 4.2: Find the derivative of $f(x)=3x^2+x+1$ by first principle of differentiation

Solution

Given $f(x)=3x^2+x+1$

$\Rightarrow f(x+h)=3(x+h)^2+(x+h)+1$

From, $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$=\lim\limits_{h\to 0}\dfrac{3(x+h)^2+(x+h)+1-(3x^2+x+1)}{h}$

$=\lim\limits_{h\to 0}\dfrac{3x^2+6xh+3h^2+x+h+1-3x^2+x-1}{h}$

$=\lim\limits_{h\to 0}\dfrac{6xh+3h^2+h}{h}$

$=\lim\limits_{h\to 0}\dfrac{h(6x+3h+1)}{h}$

$=\lim\limits_{h\to 0}(6x+3h+1)$

$=6x+1$

Therefore, the derivative of $f(x)=3x^2+x+1$ is $=6x+1$

Example 4.3: Differentiate $f(x)=\sqrt{x}$ by using the first principle.

Solution

Given $f(x)=\sqrt{x}$ then, $f(x+h)=\sqrt{x+h}$

$f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$=\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

We rationalize the numerator to get,

$f'(x)=\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}\times\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$=\lim\limits_{h\to 0}\dfrac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$

$=\lim\limits_{h\to 0}\dfrac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$

$=\lim\limits_{h\to 0}\dfrac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$

$=\lim\limits_{h\to 0}\dfrac{1}{\sqrt{x+h}+\sqrt{x}}=\dfrac{1}{2\sqrt{x}}$

Therefore, $f'(x)=\dfrac{1}{2\sqrt{x}}$

Exercise 4.1

Use the first principle of differentiation to find the derivatives of the following:

1. $f(x)=2x-3$
2. $f(x)=3x^2-4x$
3. $g(x)=ax^2+bx+c$
4. $h(x)=1-3x^3$
5. $f(x)=x^3+6x^2+4x-2$

Differentiate the following functions by using first principle of differentiation.

6. $f(x)=x(x+3)^2$
7. $f(x)=\dfrac{(x+2)(3x+1)}{5}$
8. $f(x)=\dfrac{x^2-2x^3}{5x}$
9. $f(x)=\dfrac{1}{x+1}$
10. $f(x)=\sqrt{2x+1}$

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The derivatives of $f(x)=x^n$

Activity 4.1

Consider the power function, $f(x)=x^r$, where r is an integer. Using first principle of differentiation, find the derivatives of $f(x)=x^r$, when $r=0,\ 1,\ 2$ and 3

From the results in activity 4.1, it can be deduced that if $r=n$, then $y=x^n$ and this implies that $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$

Therefore, the general formula for the derivative of the function $f(x)=x^n$ is given by:

$$f'(x)=\frac{d}{dx}(f(x))=\frac{d}{dx}(x^n)=nx^{n-1}$$

4.2 Techniques of differentiation

Derivatives of polynomial functions

The derivatives of polynomial functions multiplied by a scalar and the derivatives of sum of the functions can be obtained as follows:

a) If $f(x)=x^n$, then $\frac{d}{dx}(x^n)=nx^{n-1}$

b) If $f(x)=c$ and c is constant or any variable apart from x, then its derivative is zero, that is $f'(x)=\frac{d}{dx}(c)=0$

c) If k is any scalar and f is a differentiable function, then

$$\frac{d}{dx}(kf(x))=k\frac{d}{dx}(f(x))$$

d) If $f(x)$ and $g(x)$ are both differentiable functions; then,

$$\frac{d}{dx}\left(f(x)\pm g(x)\right)=\frac{d}{dx}\left(f(x)\right)\pm\frac{d}{dx}\left(g(x)\right)$$

Activity 4.2

If f and g are both differentiable functions of x, then use the first principles of differentiation to show that,

a) $\frac{d}{dx}\left(f(x)\pm g(x)\right)=\frac{d}{dx}\left(f(x)\right)\pm\frac{d}{dx}\left(g(x)\right)$

b) $\frac{d}{dx}\left(kf(x)\right)=k\frac{d}{dx}\left(f(x)\right)$, where k is any scalar.

Example 4.4: Find the derivative of $f(x)=3x^2$.

Solution

Given $f(x)=3x^2$,

using $\frac{d}{dx}(x^n)=nx^{n-1}$

then, $f'(x)=\frac{d}{dx}\left(3x^2\right)$

$=3\times2\times x^{2-1}$

$=6x$

Therefore, $f'(x)=6x$

Example 4.5: Find the derivative of

$$f(x)=3x^2-5x+2$$

Solution

Using $\frac{d}{dx}(x^n)=nx^{n-1}$ we have that

$f'(x)=\frac{d}{dx}\left(3x^2-5x+2\right)$

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$$= \frac{d}{dx}\left(3x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}\left(2x^0\right)$$

$$= 3\times 2\times x^{2-1} - 5\times 1\times x^{1-1} + 2\times 0\times x^{0-1}$$

$$= 6x - 5$$

Therefore, $f'(x) = 6x - 5$

Example 4.6: Find the derivative of

$$f(x) = Ax^5 + Bx + C$$

Solution

Given $f(x) = Ax^5 + Bx + C$

Using $\frac{d}{dx}(x^n) = nx^{n-1}$ then,

$$f'(x) = \frac{d}{dx}\left(Ax^5 + Bx + C\right)$$

$$= \frac{d}{dx}\left(Ax^5\right) + \frac{d}{dx}(Bx) + \frac{d}{dx}(C)$$

$$= 5Ax^{5-1} + 1\times Bx^{1-1} + 0\times C$$

$$= 5Ax^4 + B$$

Therefore, $f'(x) = 5Ax^4 + B$

Example 4.7: By using the rule of differentiation, find the derivative of $f(x) = 6x^4 - x^3 + x - 4$.

Solution

Using $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$ we get,

$$f'(x) = \frac{d}{dx}(6x^4 - x^3 + x - 4)$$

$$= 4\times 6x^{4-1} - 3\times x^{3-1} + 1\times x^{1-1} - 0\times 4x^{0-1}$$

$$= 24x^3 - 3x^2 + x^0 - 0$$

$$= 24x^3 - 3x^2 + 1$$

$$\therefore f'(x) = 24x^3 - 3x^2 + 1$$

Exercise 4.2

Find the derivative of each of the following:

1. $f(x) = -4x^2 + 6x$
2. $f(x) = x^3 - 2x^2 - 5x - 4$
3. $f(x) = (x+3)(-2x+1)$
4. $g(x) = \frac{1}{x^6} + 3x^2$
5. $g(x) = 4\sqrt{x}\left(1 - \sqrt{x}\right)$
6. $g(x) = \frac{x^2 + 4x + 3}{x^3}$
7. $h(x) = \frac{2x+4}{\sqrt{x}}$
8. $h(x) = \frac{1+x^2}{x} + 4x$
9. $h(x) = \frac{\sqrt{x}+1}{\sqrt{x}}$
10. $f(x) = 1 + \frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6}$

The Chain Rule of Differentiation

The chain rule is a rule for differentiating compositions of two or more functions.

Suppose $y = f(u)$ where $u = g(x)$ then;

$\frac{dy}{dx} = f'(u)$ and $\frac{du}{dx} = g'(x)$. This implies that;

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = f'(u)\times g'(x)$$

Therefore, the chain rule is given by;

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$

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Example 4.8: Differentiate $y=(4-x)^2$ with respect to x.

Solution

Given that $y=(4-x)^2$

Let $u=4-x$ which implies that $y=u^2$

Then, $\frac{du}{dx}=-1$ and $\frac{dy}{du}=2u$,

By chain rule, we have that,

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx} \text{ then,}$$

$2u(-1)=-2u$ but

$-2(4-x)=2x-8$

$$\therefore \frac{dy}{dx}=2x-8$$

Example 4.9: Differentiate

$y=(-x^2+2x+4)^{32}$ with respect to x.

Solution

Given $y=\left(-x^2+2x+4\right)^{32}$

Changing the variables,

let $u=-x^2+2x+4$.

This gives, $y=u^{32}$. Differentiation gives $\frac{du}{dx}=-2x+2$ and $\frac{dy}{du}=32u^{31}$

By chain rule, we have $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$=32u^{31}(-2x+2)$

$=32(-x^2+2x+4)^{31}(-2x+2)$

$$\therefore \frac{dy}{dx}=32(-x^2+2x+4)^{31}(-2x+2)$$

Example 4.10: Differentiate $y=\sqrt{(2x+2)^3}$ with respect to x.

Solution

Let $u=2x+4$, then $\frac{du}{dx}=2$,

So $y=u^{\left(\frac{3}{2}\right)}$ and $\frac{dy}{du}=\frac{3}{2}\left(u^{\frac{1}{2}}\right)$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=\frac{3}{2}\left(u^{\frac{1}{2}}\right)\times 2$$

$$=3\sqrt{(2x+4)}$$

$$\therefore \frac{dy}{dx}=3\sqrt{(2x+4)}$$

Exercise 4.3

Find the derivative of each of the following:

1. $f(x)=(3+2x)^3$
2. $g(x)=\frac{1}{2x^2+4}$
3. $h(x)=\frac{1}{\sqrt{2x-1}}$
4. $g(v)=\frac{1}{\sqrt{1+v^3}}$
5. $f(z)=\frac{1}{\left(z^3-4z\right)^2}$
6. $f(x)=4\left(2-x^3\right)^{-6}$
7. $h(x)=\frac{5}{\left(3+x^2\right)^{\frac{3}{2}}}$

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8. $f(x)=\dfrac{-1}{\sqrt[3]{1+x}}$ 9. $g(x)=\dfrac{1}{\left(x^2-\sqrt{x}\right)^2}$ 10. $h(x)=\left(3x^3-\dfrac{1}{x}\right)^{\frac{5}{3}}$

The product rule of differentiation

Consider $y=uv$ where $u=f(x)$ and $v=g(x)$ are differentiable functions, then the product rule is given as, $\dfrac{dy}{dx}=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$.

Example 4.11: Find the derivative of $y=\left(x^3+5x+2\right)\left(5x^2-3x\right)$.

Solution

Let $u(x)=\left(x^3+5x+2\right)$ and $v(x)=\left(5x^2-3x\right)$

By product rule,

$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}=\left(5x^2-3x\right)\frac{d}{dx}\left(x^3+5x+2\right)+\left(x^3+5x+2\right)\frac{d}{dx}\left(5x^2-3x\right)$$

$$=\left(5x^2-3x\right)\left(3x^2+5\right)+\left(x^3+5x+2\right)(10x-3)$$

$$=15x^4+25x^2-9x^3-15x+10x^4-3x^3+50x^2-15x+20x-6$$

$$=25x^4-12x^3+75x^2-10x-6$$

$$\therefore\frac{dy}{dx}=25x^4-12x^3+75x^2-10x-6$$

Example 4.12: Differentiate $f(x)=(x+4)\left(4x^3+2x^2\right)$ with respect to x.

Solution

Let $u=(x+4)$ and $v=\left(4x^3+2x^2\right)$

By using product rule, $\dfrac{dy}{dx}=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$

$$\Rightarrow\frac{df}{dx}=\left(4x^3+2x^2\right)\frac{d}{dx}(x+4)+(x+4)\frac{d}{dx}\left(4x^3+2x^2\right)$$

$$=\left(4x^3+2x^2\right)(1)+(x+4)\left(12x^2+4x\right)$$

$$=16x^3+54x^2+16x$$

$$\therefore f'(x)=16x^3+54x^2+16x$$

Example 4.13: Find the derivative of $y=x^2\sqrt{x^3-5}$

Let $u=x^2\Rightarrow\dfrac{du}{dx}=2x$ and

$$v=\sqrt{x^3-5}\Rightarrow\frac{dv}{dx}=\frac{d}{dx}\left(\sqrt{x^3-5}\right).$$

Using chain rule, $\dfrac{dv}{dx}=\dfrac{3x^2}{2\sqrt{x^3-5}}$

By product rule, we have

$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$

$$\frac{dy}{dx}=\left(\sqrt{x^3-5}\right)(2x)+x^2\left(\frac{3x^2}{2\sqrt{x^3-5}}\right)$$

$$\therefore\frac{dy}{dx}=\frac{7x^4-20x}{2\sqrt{x^3-5}}$$

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Exercise 4.4

Use the product rule to differentiate with respect to x each of the following.

1. $y=2x(4x-3)$
2. $y=x^2(1-x)^2$
3. $y=(1-x^2)(x+3)^2$
4. $y=(x+4)^2(1+x)^3$
5. $y=\sqrt{(x+4)(1-x^2)}$
6. $y=(1+2x)\sqrt{1-4x}$
7. $y=(x^2-1)(x+2)^2$
8. $y=(2x-1)^2(3x+5)^4$
9. $y=x^2\left(3-\frac{1}{\sqrt{2x}}\right)$
10. $y=(x^2+1)\sqrt{3-4x^2}$

The quotient rule of differentiation

Consider $y=\frac{u}{v}$ where $u=f(x)$ and $v=g(x)$ are differentiable functions, then the quotient rule is given by;

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2} \text{ or } y'=\frac{vu'-uv'}{v^2}$$

Example 4.14: Differentiate $y=\frac{3x}{2x+3}$ with respect to x.

Solution

Given $y=\frac{3x}{2x+3}$

Let $u=3x$ and $v=2x+3$. These gives; $\frac{du}{dx}=3$ and $\frac{dv}{dx}=2$

Using the quotient rule;

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$=\frac{(2x+3)\frac{d}{dx}(3x)-3x\frac{d}{dx}(2x+3)}{(2x+3)^2}$$

$$=\frac{(2x+3)(3)-3x(2)}{(2x+3)^2}$$

$$=\frac{6x+9-6x}{(2x+3)^2}$$

Therefore, $\frac{dy}{dx}=\frac{9}{(2x+3)^2}$

Example 4.15: Differentiate the following functions with respect to x.

a) $y=\frac{4x+2}{3x^2+2x}$

b) $y=\frac{x^2-4}{\sqrt{x^2+4}}$

Solution

a) Given $y=\frac{4x+2}{3x^2+2x}$

Let $u=4x+2$ and $v=3x^2+2x$

It implies that $\frac{du}{dx}=4$ and,

$\frac{dv}{dx}=6x+2$

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From $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx}=\frac{\left(3x^2+2x\right)\frac{d}{dx}(4x+2)-(4x+2)\frac{d}{dx}\left(3x^2+2x\right)}{\left(3x^2+2x\right)^2}$$

$$=\frac{\left(3x^2+2x\right)(4)-(4x+2)(6x+2)}{\left(3x^2+2x\right)^2}$$

$$=\frac{-12x^2-12x-4}{(3x^2+2x)^2}=\frac{-4\left(3x^2+3x+1\right)}{x^2(3x+2)^2}$$

$$\therefore \frac{dy}{dx}=\frac{-4\left(3x^2+3x+1\right)}{x^2\left(3x+2\right)^2}$$

b) Given $y=\frac{x^2-4}{\sqrt{x^2+4}}$

Let $u=x^2-4$ so that $\frac{du}{dx}=2x$

$v=\sqrt{x^2+4}$

So that $\frac{dv}{dx}=\frac{d}{dx}\left(\sqrt{x^2+4}\right)$

Using chain rule;

$$\frac{dv}{dx}=\frac{x}{\sqrt{x^2+4}}$$

Apply quotient rule, $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx}=\frac{2x\sqrt{x^2+4}-(x^2-4)\frac{x}{\sqrt{x^2+4}}}{(x^2+4)}$$

$$\therefore \frac{dy}{dx}=\frac{x^3+12x}{\left(x^2+4\right)\sqrt{\left(x^2+4\right)}}$$

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Exercise 4.5

Differentiate each of the following with respect to x using the quotient rule:

1. $y=\dfrac{3}{(1+2x)^2}$
2. $y=\dfrac{x}{x+1}$
3. $y=\dfrac{x^2-4}{3-x}$
4. $y=\dfrac{1+x^2}{(2+4x)^2}$
5. $y=\sqrt{\dfrac{1}{1+2x}}$
6. $y=\dfrac{(x+3)^2}{(1-3x)^2}$
7. $y=\sqrt{\dfrac{x+4}{x-4}}$
8. $y=\dfrac{2+\sqrt{2x}}{2-\sqrt{2x}}$
9. $y=\dfrac{x}{\sqrt{x+4}}$
10. $y=\dfrac{x^2}{\sqrt{3+2x^2}}$

4.3 First and second derivatives of polynomial functions

The derivative of function $y=f(x)$ is denoted by $y=f'(x)$ or $\dfrac{dy}{dx}$ and is known as the first derivative. The second derivative of the function $y=f(x)$ is obtained by differentiating the first derivative and is denoted by $f''(x)$ or $\dfrac{d^2y}{dx^2}$.

So, $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ or $f''(x)=\dfrac{d}{dx}\left(f'(x)\right)$

Example 4.16: Find the second derivative of the following:

a) $f(x)=2x^4$ b) $f(x)=x^7$

c) $f(x)=x^3-x$

Solution

a) Given $f(x)=2x^4$

$f'(x)=4\times 2x^3=8x^3$ then,

$f''(x)=\dfrac{d}{dx}(8x^3)=3\times 8x^2$

Therefore, $f''(x)=24x^2$

b) Given $f(x)=x^7$

$f'(x)=7\times x^6=7x^6$ then,

$f''(x)=\dfrac{d}{dx}(7x^6)=6\times 7x^5$

Therefore $f''(x)=42x^5$

c) Given $f(x)=x^3-x$

Taking differentiation, with respect to x gives;

$f''(x)=\dfrac{d}{dx}(3x^2-1)=2\times 3x=6x$

Therefore, $f''(x)=6x$

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Exercise 4.6

Find the second derivatives of each of the following:

1. $f(x)=3x^3$
2. $f(x)=x^3-3x^2+7$
3. $f(x)=ax^n$
4. $f(x)=-4x+3x^2-4x^3$
5. $f(x)=(2x+4)^3$
6. $f(x)=x(1-x^2)^2$
7. $f(x)=2x^3(1+x^2)^2$
8. $f(x)=3x\sqrt{1-x}$
9. $f(x)=\sqrt[3]{x+2}$
10. $f(x)=\dfrac{x}{1+x}$

4.4 Implicit differentiation

If the dependent variable of a function is expressed in terms of the independent variable such as $y=f(x)$, then the function is said to be an explicit function. The following are examples of explicit functions.

a) $y=2x^2+3x+7$

b) $y=\dfrac{6x+1}{x^2+3}$

Implicit function is a function of the form $f(x, y, z,...)=0$ It is the function whose variables are connected by an equation which does not give one variable in terms of the other. The following are examples of implicit functions:

a) $2x^2+xy^2+y^3=12$

b) $yx^2+y^2x^2+4xy=0$

c) $\dfrac{x^2}{16}+\dfrac{y^2}{2x^2}=5xy$

d) $x^3-y^3=xy$

In order to find the derivative of an implicit function, differentiate each term with respect to the identified independent variable given.

Example 4.17: Find $\dfrac{dy}{dx}$ for the following:

a) $2x+y^3=xy$

b) $x^2y^2+xy^3=k$, where k is a constant

c) $x^3+xy^2-4y^3=5$

d) $3x^2-7y^2+4xy-8x=0$ at point $(-1,1)$

Solution

a) Given $2x+y^3=xy$. Taking differentiation with respect to all the variables gives,

$$2dx+3y^2dy=ydx+xdy$$

Divide $2dx+3y^2dy=ydx+xdy$ through out by dx to get

$$\frac{2dx}{dx}+3y^2\frac{dy}{dx}=\frac{ydx}{dx}+\frac{xdy}{dx}$$

$$2+3y^2\frac{dy}{dx}=y+\frac{xdy}{dx}$$

Rearranging the terms we get

$$(3y^2-x)\frac{dy}{dx}=y-2$$

$$\therefore \frac{dy}{dx}=\frac{y-2}{3y^2-x}$$

b) Given

$x^2y^2+xy^3=k$ where k is a constant

Taking derivatives of all the variables gives,

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$2xy^2dx+2x^2ydy+y^3dx+3xy^2dy=0$

Divide by dx through out the equation to get

$$2xy^2dx+2x^2y\frac{dy}{dx}+y^3dx+3xy^2\frac{dy}{dx}=0$$

Rearranging the equation above we get

$$\left(2x^2y+3xy^2\right)\frac{dy}{dx}=-2xy^2-y^3$$

This implies that $\frac{dy}{dx}=\frac{-2xy^2-y^3}{2x^2y+3xy^2}$

which simplifies to $\frac{dy}{dx}=\frac{-y(y+2x)}{x(2x+3y)}$

c) Given that $x^3+xy^2-4y^3=5$

Take derivative of all the variables to get

$3x^2dx+y^2dx+2xydy-12y^2dy=0.$

Divide by dx through out the equation to get

$$3x^2dx+y^2dx+2xy\frac{dy}{dx}-12y^2\frac{dy}{dx}=0.$$

Rearranging the equation above gives

$$\left(2xy-12y^2\right)\frac{dy}{dx}=-3x^2-y^2$$

$$\therefore\frac{dy}{dx}=\frac{-3x^2-y^2}{2y(x-6y)}$$

d) Given $3x^2-7y^2+4xy-8x=0$ at point $(-1,1)$. First, differentiate with respect to all the variables as follows;

$6xdx-14ydy+4ydx+4xdy-8dx=0.$

Divide by dx throughout the equation

$$6x-14y\frac{dy}{dx}+4ydx+4x\frac{dy}{dx}-8=0.$$

Rearranging the terms gives

$$(4x-14y)\frac{dy}{dx}=8-4y-6x$$

$$\frac{dy}{dx}=\frac{4-2y-3x}{2x-7y}$$

At the point $(-1, 1)$

$$\left.\frac{dy}{dx}\right|_{(-1,1)}=\frac{4-2(1)-3(-1)}{2(-1)-7}$$

$$=\frac{4-2+3}{-2-7}=\frac{5}{-9}$$

$$\therefore\frac{dy}{dx}(-1,1)=\frac{-5}{9}$$

Exercise 4.7

Using implicit differentiation, find $\frac{dy}{dx}$ for each of the following equations:

1. $y=(x+y)^2$
2. $xy^2+x^5y+4y^3=13x^2$
3. $x^2-2y^4=18xy$
4. $xy^5=10x^2-8y$
5. $y^2=xy-3x^2$
6. $xy^4-2y=x^3$
7. $y^3x^2+8xy=4$
8. $x^3+3x^2y+3xy^2+y^3=0$
9. $x^2+y^3-y+x^{-2}=0$
10. $x^2+2xy-2y^2+2x=10$

4.5 Applications of differentiation

Differentiation can be applied in many aspects. Some of the applications of differentiation are realised when finding slope of curves, rate of change, critical points of a curve, marginal cost, and marginal revenue.

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Slope of a curve at a given point

The slope of a curve at a given point is defined as the slope of the tangent to the curve at the point and measures the rate of increase of y with respect to x.

The slope of the curve is always equal to the slope of the tangent at the point of contact.

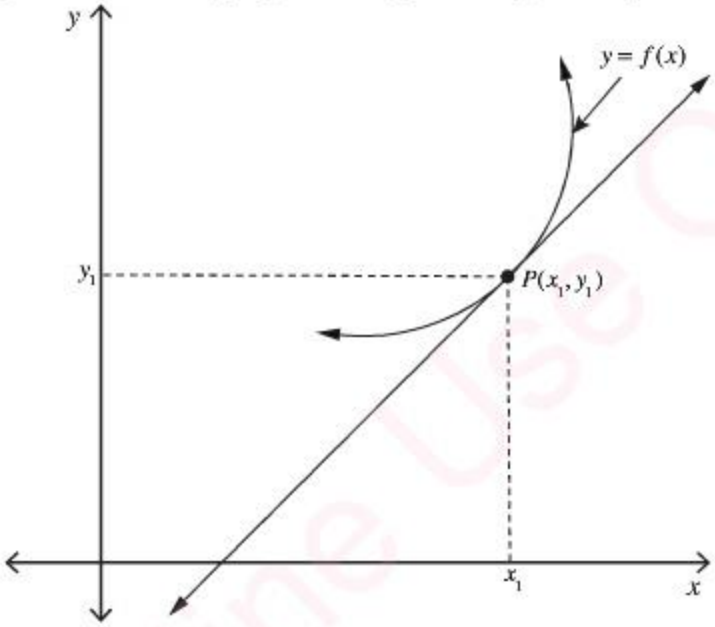

Figure. 4.2 Illustration showing the slope of a curve at a given point

From Figure 4.2, slope $m=\frac{dy}{dx}\left(\text{at } x=x_1\right)=f'\left(x_1\right)$ at the point $P\left(x_1,y_1\right)$

Example 4.18: Find the slope of the curve $y=x^2$ at the point (4,16).

Solution

Given the curve $y=x^2$, its slope is given by

slope, $m=\frac{dy}{dx}$

Differentiation with respect to x gives,

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2\right)=2x$$

At the point (4,16) $\frac{dy}{dx}=2\times4=8$

Therefore, slope of the curve $y=x^2$ at (4,16) is 8.

Example 4.19: Calculate the gradient of the tangent to:

a) $y=x^2-2x+1$ at $x=2$

b) $y=(x+2)(x-4)$ at $x=3$

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Solution

a) Given $y=x^2-2x+1$

Gradient $m=\frac{dy}{dx}$

Then, $m=\frac{d}{dx}\left(x^2-2x+1\right)$

$=2x-2$

at $x=2$, $\frac{dy}{dx}=2\times2-2=2$

Therefore, the gradient of the curve $y=x^2-2x+1$ at $x=2$ is 2.

b) Given $y=(x+2)(x-4)$

Gradient, $m=\frac{dy}{dx}$

$m=\frac{d}{dx}\left[(x+2)(x-4)\right]$

$=\frac{d}{dx}\left(x^2-2x-8\right)$

$=2x-2$

at $x=3$, $\frac{dy}{dx}=2\times3-2$

Therefore, the gradient of the curve $y=(x+2)(x-4)$ at $x=3$ is 4.

Exercise 4.8

In question 1 to 5, find the gradient of the given curve at the point indicated.

1. $y=x^2-2$ at $x=1$
2. $y=(x+2)(x-2)$ at $x=-\frac{1}{2}$
3. $y=\sqrt{x^2+1}$ at $x=0$
4. $y=\frac{1}{x^2}$ at $x=-2$
5. $6(x-y)=4xy+2$ at $x=3$
6. Find the coordinates of the point on the curve $y=2x^3-10$ at which its gradient is 6.
7. The slope of the curve $y=x^2+ax+1$ at $x=2$ is 7. Find the value of a.
8. Find the slope of the curve $y=x^2+4x+4$ at the point where the curve cuts the $y-$axis.
9. Determine the slope of the curve $y=x^2+5x-2$ at the point where the curve cuts the line $x=1$.
10. Find the value of p for which the line $y=3x+p$ is tangent to the curve $y=x^2-5$.

Rates of change

The word rate refers to the change of substance or quantity per unit time. Understanding the nature of change and the rate at which it takes place enables the experts to make important predictions and decisions about the atmospheric pressure, humidity, wind patterns, temperature, and others. Differentiation help us to determine the rate of change of quantities such as volumes, populations, areas, lengths, and motion.

The chain rule can be used when dealing with problems involving rates of change because, some problems involving rate

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of change of one variable compared with another do not provide a direct relationship between those variables. The following examples show direct application of differentiation in real life situations.

Example 4.20: The area of the circle is increasing at the rate of 6 $cm^2 s^{-1}$. Find the rate of change of radius of the circle when the radius of the circle is 4 cm.

Solution

If at time t, the area of the circle is A and its radius is r, then

$$A = \pi r^2$$

Taking differentiation of A with respect to r gives

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

Given that, $\frac{dA}{dt} = 6\ cm^2/s$

Applying the chain rule to find $\frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt} \text{ but, } \frac{dr}{dA} = \frac{1}{\frac{dA}{dr}}$$

$$\frac{dr}{dt} = \frac{1}{2\pi r} \times 6 = \frac{3}{\pi r}$$

When $r = 4$ cm, then

$$\frac{dr}{dt} = \frac{3}{\pi \times 4} = \frac{3}{4\pi}\ cms^{-1}$$

Therefore, the rate of increase of radius is $\frac{3}{4\pi}\ cms^{-1}$, when the radius is 4 cm.

Example 4.21: A spherical balloon is blown up so that its volume increases at a constant rate of $1cm^3s^{-1}$. Find the rate of increase of the radius when the volume is $6cm^3$.

Solution

If at time t, the volume and radius of the balloon are given by V and r respectively, then;

$$V = \frac{4}{3}\pi r^3$$

Differentiation of V with respect to r gives,

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$$

Applying the chain rule,

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$\frac{dr}{dt} = \frac{dV}{dt} \div \frac{dV}{dr} = \frac{1}{4\pi r^2}$$

Given $V = 6cm^3$, using $V = \frac{4}{3}\pi r^3$

gives $r = 1.13$cm.

$$\frac{dr}{dt} = \frac{1}{4\pi(1.13)^2} = \frac{1}{5.11\pi} = 0.062\ cms^{-1}$$

Therefore, the radius is increasing at $0.062\ cms^{-1}$ when the volume is $6cm^3$.

Example 4.22: Water runs into a conic tank at the rate of $9m^3min^{-1}$. The tank slants point down and has a height of 10 m and a base radius of 5 m. How fast is the water level rising when the water is 6 m deep?

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Solution

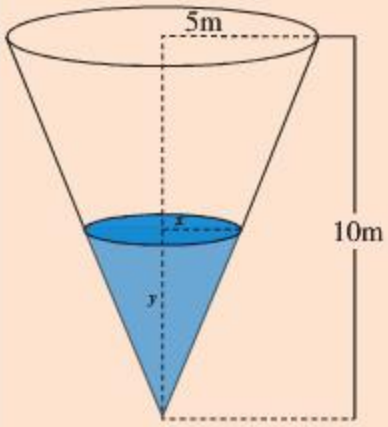

Figure 4.3 A conic tank

Let;

V represent the volume of water in the tank at time t in cubic metres.

x be the radius of surface water at time t in metres.

y be the depth of water in the tank at time t in metres.

If V, x and y are differentiable functions of t; then $\frac{dy}{dt}$ is the rate at which water level rises with time.

Given $\frac{dV}{dt} = 9 \text{ m}^3/\text{min}$

Volume of the cone $V = \frac{1}{3}\pi x^2 y$ where $r = x$ and $h = y$

Similar triangles gives $\frac{y}{x} = \frac{10}{5} = 2$.

It implies that, $x = \frac{y}{2}$

$$\therefore V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

Then $\frac{dV}{dt} = \frac{dV}{dy} \times \frac{dy}{dt}$ but $\frac{dV}{dy} = \frac{\pi}{4}y^2$

$$\Rightarrow \frac{dV}{dt} = \frac{\pi}{4}y^2 \times \frac{dy}{dt}$$

Substituting $y = 6$ and $\frac{dV}{dt} = 9$ yields

$$9 = \frac{\pi}{4}(6)^2\frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{\pi} \text{ m/min}$$

Therefore, the water level is rising at the rate of $\frac{1}{\pi}$ m / min.

Exercise 4.9

1. If $y = x + x^2$, find $\frac{dx}{dy}$ when $x = 2$ and $\frac{dy}{dx}$ when $x = 1$.
2. The volume of the cube increases at the rate of 30 cm³ / s. Find the rate of increase of surface area when the length of a side is 12 cm.
3. The surface area of spherical balloon decreases at the rate of 0.3 cm² / s when the radius is 0.2 cm. Calculate the rate of change of the volume of the spherical balloon at this instance.
4. Ink is dropped on the blotting paper forming a circular stain which increases at the rate of 2.5 mm² / s. Find the rate of change of circumference when the area is 15 mm².
5. A container in the shape of a right circular cone of height 20 cm and base radius 3 cm is catching the drips from a tap leaking at the rate of 0.4 cm³ / s. Find the rate at which the water level is raising when the water level is half-way up the cone.

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Displacement, velocity, and acceleration of a moving object

The concept of rates of change can be used in motion of objects to find velocity and acceleration when the displacement function is known.

Consider the function $s(t)$ representing the position of an abject moving in a straight line at any time t. The velocity $v(t)$ is the rate of change of displacement; i.e

$v(t)=\frac{ds}{dt}=s'(t)$, and acceleration $a(t)$ is the rate of change of velocity; i.e.

$a(t)=\frac{dv}{dt}=v'(t)$

Example 4.23: Suppose a ball is thrown straight up in the air such that its position is described by the expression, $s(t)=50t-5t^2$ in metres. Find:

a) Its velocity after 3 seconds.

b) Its acceleration.

Solution

a) Given that $s(t)=50t-5t^2$. Then,

velocity, $v(t)=\frac{ds}{dt}$

$=\frac{d}{dt}\left(50t-5t^2\right)$

$v(t)=50-10t$

When $t=3$ sec, $v=50-10(3)$

$v(t)=20$ m / s

$\therefore$ Velocity $=20$ m / s

b) Acceleration,

$a(t)=\frac{dv}{dt}$

$=\frac{d}{dt}\left(50-10t\right)$

$=-10$ m / s^2

$\therefore$ Acceleration $=-10$ m / s^2

Example 4.24: An object travels such that its position is described by the function $s(t)=3+10t-3t^2$. Find the velocity and acceleration functions.

Solution

Given $s(t)=3+10t-3t^2$

From $v(t)=\frac{ds}{dt}=\frac{d}{dt}\left(3+10t-3t^2\right)$

$v(t)=10-6t$

Therefore the velocity function is,

$v(t)=10-6t$.

Acceleration $a(t)=\frac{dv}{dt}$

$=\frac{d}{dt}(10-6t)$

$=-6$ m / s^2

Therefore the acceleration function is

$a(t)=-6$ m / s^2

Activity 4.3

A particle moves back and forth along a horizontal line defined by the position function $s(t)=t^3-12t^2+36t-30$, for $t\geq 0$.

a) Determine the velocity and acceleration function.

b) When is the velocity zero? When is the acceleration zero?

c) At what time interval is the velocity negative? Explain its physical meaning.

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d) When is the particle speeding up and slowing down?

e) Describe the motion of the particle.

Exercise 4.10

1. If $s(t)=3t+t^3$ is a function of displacement in metres, calculate the velocity and acceleration after 2 seconds.
2. The height covered by a stone dropped from the top of a building S metres is given by $S(t)=4.9t^2+2$ at any time t in seconds. Find the velocity of the falling stone at time $t=3$ seconds.
3. A particle moving in a straight line has its displacement governed by the equation $S(t)=-2t^2+10t-1,\ t\geq 0$, where S is in metres and t in seconds. Find the velocity and acceleration of a particle at time t.
4. A particle is moving in straight line such that its position is described by the function $s(t)=100t-10t^2$ metres. When will it come to rest if t is in seconds?
5. The position function of a stone thrown from a cliff is given by $s(t)=10t-16t^2$ metres (below the cliff) after t seconds. What is the average velocity of the stone between $t_1=1$ and $t_2=5$ seconds?

Critical (stationary) points of a curve

Consider Figure. 4.4 which shows a curve in which the gradient changes from point to point along its curve which illustrates critical points.

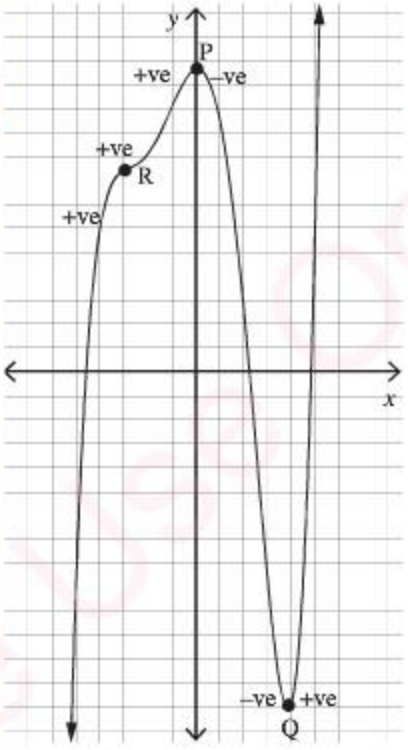

Figure. 4.4 A Curve showing critical points and its change of signs.

In Figure 4.4, the point P indicates maximum turning point where the derivative of the function of the curve is equal to zero and changes sign from positive to negative.

The point Q indicates a minimum turning point where the derivative of the function of the curve is equal to zero and changes sign from negative to positive. The point R indicates an inflexion point which is a point where the gradient of a curve is zero, but there is no change of sign for the gradient.

Differentiation can help to determine critical points and their nature. The first derivative and the test of change of signs determine the nature of the critical points.

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The first derivative followed by the second derivative can be used to test whether the critical points are maximum, minimum, or inflexion points. The following conditions are used.

a) If $f'(x)=0$ and $f''(x)>0$, then the critical point is minimum.

b) If $f'(x)=0$ and $f''(x)<0$, then the critical point is maximum.

c) If $f'(x)=0$ and $f''(x)=0$, then the critical point is an inflexion point.

Example 4.25: Determine the critical points and their nature for the curve $y=x^3+3x^2-9x$.

Solution

Given that $y=x^3+3x^2-9x$, then

$$\frac{dy}{dx}=3x^2+6x-9$$

Critical points occur when $\frac{dy}{dx}=0$

$$0=3x^2+6x-9$$

$$\Rightarrow 0=x^2+2x-3$$

$$\Rightarrow 0=(x+3)(x-1)$$

$$\therefore x=1 \text{ and } x=-3$$

Testing the left and right signs of the derivative of each value, proceed as in the following table.

x – value	left	–3	right	left	1	right
Sign of $\frac{dy}{dx}$	+	0	–	–	0	+

At $x=-3$, there is a maximum turning point since the sign of $\frac{dy}{dx}$ changes from positive to negative.

Then, $y_{max}=(-3)^3+3(-3)^2-9(-3)$

$$y_{max}=(-3)^3+3(-3)^2-9(-3)$$

$$y_{max}=27$$

Therefore, the point (–3, 27) is a maximum turning point.

At $x=1$, there is a minimum turning point since the sign of $\frac{dy}{dx}$ changes from negative to positive.

Then $y_{min}=(1)^3+3(1)^2-9(1)$

$$y_{min}=-5$$

Therefore, the point (–3, 27) is a maximum turning point.

Alternatively;

From $y=x^3+3x^2-9x$

$$\Rightarrow \frac{dy}{dx}=3x^2+6x-9$$

Critical points occur when $\frac{dy}{dx}=0$. That is,

$$0=3x^2+6x-9$$

$$\Rightarrow 0=x^2+2x-3$$

$$\Rightarrow 0=(x+3)(x-1)$$

$$\therefore x=1 \text{ and } x=-3$$

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Calculating the second derivative;

From $\frac{dy}{dx} = 3x^2 + 6x - 9$

$\Rightarrow \frac{d^2y}{dx^2} = 6x + 6$

when $x = -3$, gives

$\frac{d^2y}{dx^2} = 6(-3) + 6$

$\frac{d^2y}{dx^2} = -12$

Since the second derivative is negative, then there is a maximum turning point at $x = -3$.

Then, $y_{max} = (-3)^3 + 3(-3)^2 - 9(-3)$

$y_{max} = 27$

Therefore, the point $(-3, 27)$ is a maximum turning point.

when $x = 1$, Then,

$\frac{d^2y}{dx^2} = 6(1) + 6 = 12$

$\frac{d^2y}{dx^2} = 12$

Since the second derivative is positive, then there is a minimum turning point at $x = 1$.

That is, $y_{min} = (1)^3 + 3(1)^2 - 9(1)$

$y_{min} = -5$

$\therefore$ Minimum point $(1, -5)$

Therefore, the point $(1, -5)$ is a minimum turning point.

Example 4.26:

Find and classify the nature of all the critical points of the curve given by $y = 2x^3 + 3x^2 - 12x + 7$.

Solution

Given $y = 2x^3 + 3x^2 - 12x + 7$.

Then, $\frac{dy}{dx} = 6x^2 + 6x - 12$

For critical points, $\frac{dy}{dx} = 0$

$\Rightarrow 0 = 6x^2 + 6x - 12$

$\Rightarrow 0 = x^2 + x - 2$

$\Rightarrow 0 = (x - 1)(x + 2) = 0$

$\therefore x = 1, \ x = -2$

To test the sign of $\frac{dy}{dx}$ in the left and right of $x = 1$ and $x = -2$, study the table below.

x – value	left	-2	right	left	1	right
Sign of $\frac{dy}{dx}$	+	0	–	–	0	+

At $x = -2$, there is a maximum turning point since the sign of $\frac{dy}{dx}$ changes from positive to negative.

Then, $y_{max} = 2(-2)^3 + 3(-2)^2 - 12(-2) + 7$

$y_{max} = 27$

Therefore, the point $(-2, 27)$ is the maximum turning point.

At $x = 1$, there is a minimum turning point

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since the sign of $\frac{dy}{dx}$ changes from negative to positive .

So, $y_{min} = 2(1)^3 + 3(1)^2 - 12(1) + 7 = 0$

$y_{max} = 0$

Therefore, the point (1, 0) is a minimum turning point.

Alternatively;

From $y = 2x^3 + 3x^2 - 12x + 7$

$$\frac{dy}{dx} = 6x^2 + 6x - 12$$

For critical points, $\frac{dy}{dx} = 0$. That is,

$0 = 6x^2 + 6x - 12$

$0 = x^2 + x - 2$

$\Rightarrow = (x-1)(x+2) = 0$

$\therefore x = 1, \quad x = -2$

For the second derivative;

$$\frac{d^2y}{dx^2} = 12x + 6$$

At $x = 1$, $\frac{d^2y}{dx^2} = 12(1) + 6 = 18$

Since the second derivative is positive, then there is a minimum turning point at $x = 1$. Then,

$y_{min} = 2(1)^3 + 3(1)^2 - 12(1) + 7 = 0$

$\Rightarrow y_{max} = 0$

Therefore, the point $(1,0)$ is a minimum turning point.

At $x = -2$, $\frac{d^2y}{dx^2} = 12(-2) + 6 = -18$

Since the second derivative is negative, then there is a maximum turning point at $x = -2$

Then,

$y_{max} = 2(-2)^3 + 3(-2)^2 - 12(-2) + 7 = 0$

$y_{max} = 27$

Therefore, (–2, 27) is a maximum turning point.

Example 4.27: Sketch the graph of $f(x) = -x^2 - 6x - 9$ using differentiation technique.

Solution

Given $f(x) = -x^2 - 6x - 9$

Determining x and y intercepts, we have;

$f(x) = 0$ when $x = -3$ and $f(x) = -9$ when $x = 0$

The curve cuts the axes at $(-3,0)$ and $(0,-9)$.

For critical (stationary) points, we have;

$f'(x) = -2x - 6$

Critical points occur when $f'(x) = 0$

$-2x - 6 = 0$

$\Rightarrow x = -3$

The sign of $f'(x)$ in the left and right of $x = -3$ is as shown in the following table.

x – value	left	–3	right
Sign of $f'(x)$	+	0	–

Thus, the point (–3, 0) is the maximum turning point, therefore the graph will open downward at this point.

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The graph of $f(x)=-x^2-6x-9$ is shown in Figure 4.5.

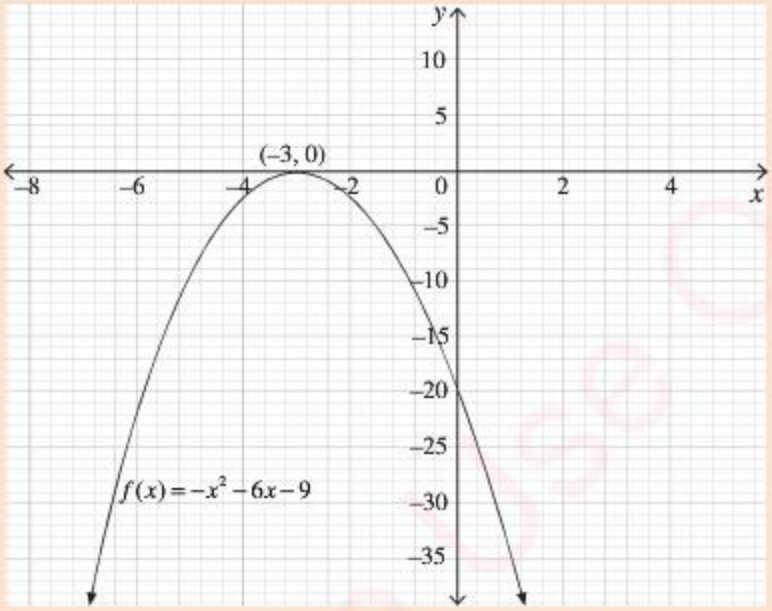

Figure 4.5 The graph of $f(x)=-x^2-6x-9$

Example 4.28: Sketch the graph of $f(x)=x^3-3x^2-9x$ using differentiation technique.

Solution

Given $f(x)=x^3-3x^2-9x$

when $f(x)=0$, the curve crosses the $x-$axis at $x=0$ and $x=-1.854$. When $x=0$, $y=0$

Taking differentiation with respect to x, we have

$$f'(x)=3x^2-6x-9$$
$$=3(x+1)(x-3)$$

For critical points, set $f'(x)=0$ and the turning points are at $x=-1$ and $x=3$

Testing the sign of $f'(x)$ in the left and right of $x=-1$

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x – value	left	–1	right
Sign of $f'(x)$	+	0	–

Thus the point $(-1, 5)$ is the maximum turning point, the graph opens downward.

Testing the sign of $f'(x)$ in the left and right of $x = 3$

x – value	left	3	right
Sign of $f'(x)$	–	0	+

Thus $(3, -27)$ is the minimum turning point, the graph opens upward.

Hence the graph of $f(x) = x^3 - 3x^2 - 9x$ is shown in Figure 4.6.

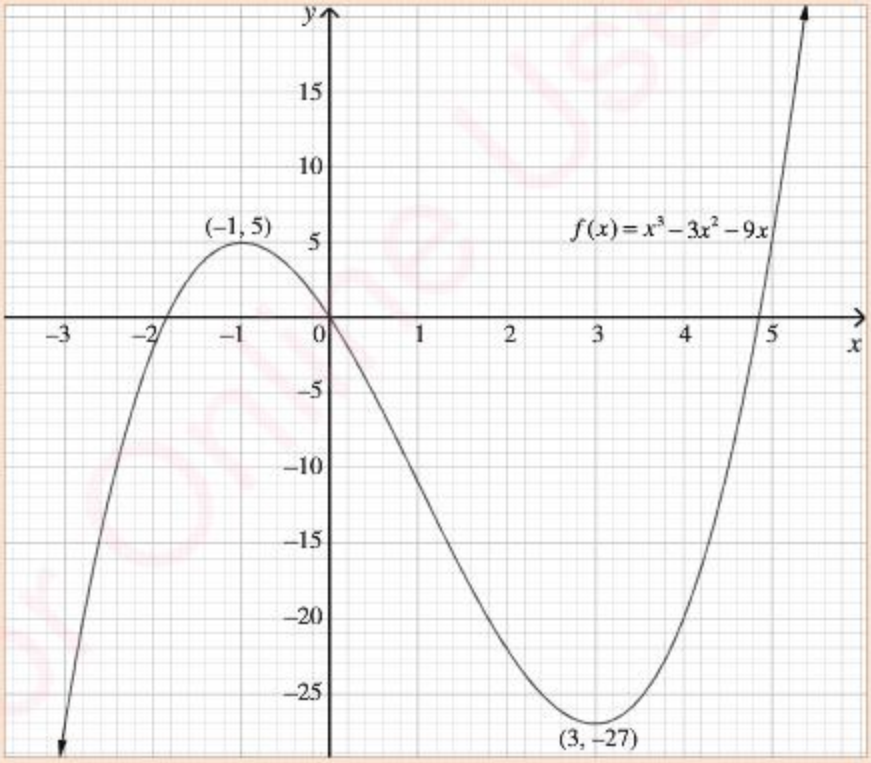

Figure 4.6 The graph of $f(x) = x^3 - 3x^2 - 9x$

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Example 4.29: A gardener has 200 m of fencing material and wishes to build a rectangular field entirely enclosed by fence. What dimensions of the field should the gardener make so as to maximize the area? Find the maximum area to be enclosed.

Solution

The field will have a length x and width y as seen in Figure 4.7.

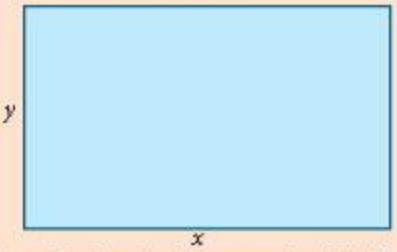

Figure 4.7. A sketch of the rectangular field with length x and width of y in metres

The area is the function that is to be maximized. It is given by $A = xy$. But the fencing material does not exceed 200 m so that the field must have maximum perimeter of 200 m.

That is, $x + x + y + y = 200$

$\Rightarrow 2x + 2y = 200 \quad \Rightarrow x + y = 100$

$\Rightarrow y = 100 - x$

Substituting $y = 100 - x$ into equation $A = xy$ yields:

$$A = x(100 - x) = 100x - x^2$$

$$\Rightarrow \frac{dA}{dx} = 100 - 2x$$

A has a stationary value when $\frac{dA}{dx} = 0$

$\Rightarrow 100 - 2x = 0$

$\Rightarrow x = 50$

when $x = 50$, $\frac{d^2A}{dx^2} = -2$.

Therefore the area is maximum when $x = 50$ and $y = 50$. That is

$$A = xy = 50 \text{ m} \times 50 \text{ m} = 2500 \text{ m}^2$$

Therefore, the gardener should make the dimension of 50 m by 50 m to enclose the maximum area of 2500 m^2.

Example 4.30: The width of a square piece of paper is 10cm on a side. A five sided box is to be made by cutting squares out of the corners and folding up the sides. What is the size of a square which must be cut out so that the volume is maximized?

Solution

The dimensions and cuttings of the square piece are shown in Figure 4.8 and 4.9, respectively. Figure 4.10 shows the resulting box. All figures are not drawn to scale.

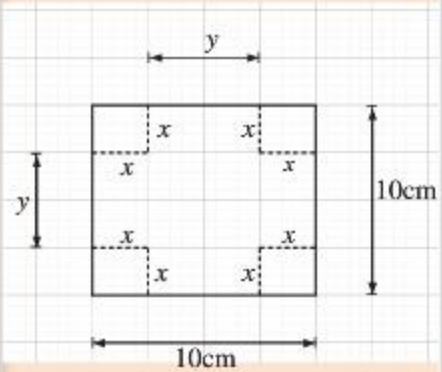

Figure 4.8 Dimensions of cuttings of small squares

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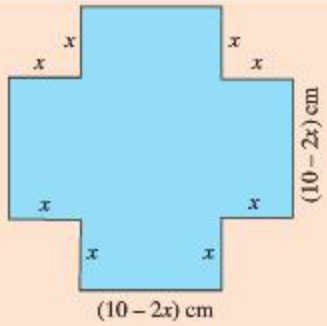

Figure 4.9 Resulting shape after removing the square corners

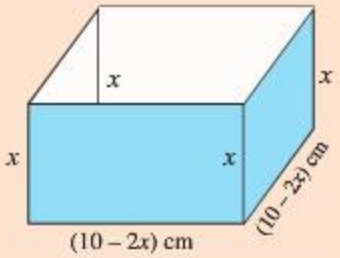

Figure 4.10 Resulting shape after removing the square corners

The volume of the box is given by the area of the base times the height. The area of the base is y^2 and the height is x. Thus, the volume is given by $V = y^2x$. Since the paper is 10 centimetre per side, then:

$$x + x + y = 10 \quad \text{to} \quad y = 10 - 2x$$

Substituting this into the equation for volume yields

$$V(x) = (10 - 2x)^2 x$$

$$= (10 - 40 \ + 4 \ ^2)$$

$$= 100 \ - 40 \ ^2 + 4 \ ^3$$

Critical points occur when $\text{——} = 0$

$$\Rightarrow \text{——} = 100 - 80 \ + 12 \ ^2$$

$$\Rightarrow 0 = 100 - 80 \ + 12 \ ^2$$

$$\Rightarrow 0 = 25 - 20 \ + 3 \ ^2$$

Solving the quadratic equation gives

$$x = 5 \text{ and } x = \frac{5}{3}$$

The second derivative test can be used to check which values give the maximum volume, that is,

$$\frac{^2}{_2} = -80 + 24$$

Now, then $\ = 5$, $\frac{^2}{_2} = -40$, minimum volume.

When $\ = \frac{5}{3}$, maximum volume.

Then, $_{\max} = \frac{5}{3}\left(10 - \frac{10}{3}\right)^2 = 74.07 \ ^3$

Therefore, the maximum volume is formed when a square of each side $\frac{5}{3}$ cm is cut at the corners.

Exercise 4.11

1. Find and classify the stationary points for each of the following:
   a) $(\ ) = 2 \ ^3 - 4 \ ^2 - 8 \ - 2$
   b) $(\ ) = \frac{}{^2 + 4}$
2. Determine and classify the nature of the critical points of each of the following:
   a) $(\ ) = 2 \ ^3 - 3 \ ^2 - 12 \ + 8$
   b) $(\ ) = \ ^3 - 3 \ ^2 + 3 \ + 7.$

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3. Sketch the graph of each of the following:
   a) $f(x)=\frac{3}{4}x^2+1$
   b) $f(x)=(x-6)(x+1)(x-1)$
4. A cardboard box with square top and bottom of equal size is to have a volume of 2000 cm³. What should be the dimensions of the box if the area of the cardboard used is as small as possible?
5. A cylindrical container with capacity 100 m³ is to be produced. The top and bottom of the container are to be made of a material that cost Tsh 600 per m² while the side of container is made of material costing Tsh 300 per m². Find the dimensions that will minimize the total cost.

Marginal cost and marginal revenue

There are three important functions in the manufacturing commodity. These are;

a) The cost function $C(x)$ which is the total cost of producing x units of the product (item) during some period of time.
b) The revenue function $R(x)$ which is total revenue obtained by selling x units of a product during some period of time
c) The profit function $P(x)$ which is the total profit obtained by selling units of product during some period of time.

The variable x is the number of items produced, the rates of change with respect to x for $C(x)$ and $R(x)$ are $\frac{d}{dx}C(x)$ and $\frac{d}{dx}R(x)$ referred to as the marginal cost and marginal revenue respectively. The word "Marginal" refers to the rate of change which resembles derivative.

Marginal cost refers to the increase in cost that accompanies a unit increase in production (output).

Let $C(x)=f(x)+k$ be a total cost function where k is a constant.

Then, the marginal cost denoted as $C'(x)$ is given by $C'(x)=\frac{d}{dx}(C(x))$

Marginal revenue refers to the extra revenue that an additional unit of product will bring to a firm.

Let total revenue function $R(x)=mx$ where m represents price per unit quantity.

Then, the marginal revenue denoted as $R'(x)$ is given by:

$$R'(x)=\frac{d}{dx}(R(x))$$

Also the profit function $P(x)$ is the difference between revenue function $R(x)$ and the cost function. That is,

$$P(x)=R(x)-C(x)$$

Example 4.31: The cost in shillings of manufacturing x units of product is given by the equation $C(x)=x^3-20x^2+300x+1000$. If each item sells for Tsh 300, find marginal cost, and marginal revenue when 1000 units are being manufactured.

Solution

Given: $C(x)=x^3-20x^2+300x+1000$

$$\text{Marginal cost}=\frac{d}{dx}\left(C(x)\right)$$

$$=\frac{d}{dx}\left(x^3-20x^2+300x+1000\right)$$

$$\therefore MC=3x^2-40x+300$$

when $x=1000$, this implies that

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$$MC(1000) = 3(1000)^2 - 40(1000) + 300$$
$$= 2{,}960{,}300$$

Therefore, the marginal cost is Tsh 2,960,300

Marginal revenue $(MR) = \frac{d}{dx}\big(R(x)\big)$, where

$$R(x) = \text{Price} \times \text{Quantity} = 300x$$
$$\therefore MR = \frac{d}{dx}(300x)$$
$$= 300$$

Therefore, the marginal revenue is Tsh 300.

Example 4.32: If a certain firm had the total cost given by the function $C(x) = 300 + 40x + 18x^2$ where x is the units of outputs, find the Marginal cost function.

Solution

Given: $C(x) = 300 + 40x + 18x^2$

$$\frac{d}{dx}\big(C(x)\big) = \frac{d}{dx}\big(300 + 40x + 18x^2\big)$$
$$\Rightarrow C'(x) = 40 + 36x$$

Therefore, the marginal cost function is $40 + 36x$

Example 4.33: The cost in Tsh of manufacturing units of a certain product is given by $C(x) = 300x - 10x^2 + \frac{1}{3}x^3$ while each unit sells for Tsh 200. If 100 units are produced, find:

a) The marginal cost

b) The marginal revenue if the total revenue is given by $R(x) = 800x + 40x^2 + x^3$

Solution

a) Given $C(x) = 300x - 10x^2 + \frac{1}{3}x^3$

Marginal cost $\frac{d}{dx}\big(C(x)\big) = C'(c)$

$$= \frac{d}{dx}\left(300x - 10x^2 + \frac{1}{3}x^3\right)$$
$$\Rightarrow C'(x) = 300 - 20x + x^2$$

when $x = 100$, we get

$$C'(100) = 300 - 20 \times 100 + 100^2 = 8\,300$$

Therefore, the marginal cost is Tsh 8,300.

b) Given $R(x) = 8000x + 40x^2 + x^3$

Marginal revenue $= \frac{d}{dx}\big(R(x)\big) = R'(x)$

$$R'(x) = \frac{d}{dx}\big(8000x + 40x^2 + x^3\big)$$
$$= 8000 + 80x + 3x^2$$

When $x = 100$, gives

$$R'(100) = 8000 + 80(100) + 3(100)^2 = 46\,000$$

Therefore, the marginal revenue is Tsh 46,000.

Exercise 4.12

1. If $C(x) = 4x + 300x^2$ is a total cost function of sales and x is the number of units sold and 2000 units were sold find,
   a) The total cost
   b) The marginal cost

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2. A process has a total cost function given by $C(x)=20+4x$ and a revenue function given by $R(x)=22x-4x^2$, find;
   a) Profit function
   b) Marginal revenue function

3. If $P(x)=\dfrac{100000}{50x-100}-50$ is a Price function for x kilograms of a certain commodity, find
   a) the Revenue function
   b) total Revenue for 10 kilograms of that commodity

4. If the total Revenue function is $R(x)=64x$ and total Cost function is $C(x)=400+4x+2x^2$ in the sell of 10 units, find the;
   a) marginal revenue
   b) marginal cost
   c) profit

5. If the Cost function for commodity is $C(x)=2x^3-8x^2+8x+40$, find the Marginal cost at $x=8$ and tell what this predict about the cost of producing one additional unit.

Revision exercise 4

1. Differentiate the following with respect to x:
   a) $f(x)=4x^3+7x+5$ b) $f(x)=4x^7$ c) $f(x)=\dfrac{4}{x^3}$

2. Find $f'(x)$ of the following:
   a) $f(x)=\dfrac{x^2-1}{x^2+1}$ b) $f(x)=\dfrac{2x-1}{x-1}$ c) $f(x)=\dfrac{3}{x+2}$

3. Find $f'(x)$ in each of the following:
   a) $f(x)=\left(2x^2+5\right)\left(x^2-1\right)$ b) $f(x)=\left(x^2-3x+1\right)\left(4x^2-1\right)$
   c) $f(x)=\sqrt{2x^2+9x+3}$ c) $f(x)=(2x-1)^2(3x+5)^4$

4. Differentiate the following with respect to x:
   a) $f(x)=\left(2x^2+7\right)$ b) $f(x)=\left(x^2+3\right)(5x+2)$
   c) $y=\dfrac{x^{-3}}{x^2-6x+3}$ c) $y=\dfrac{3-x^2}{(x-2)(x+8)}$

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5. Determine the value of $\frac{dy}{dx}$ at the given value of x:

   a) $y = x^2 + x + 1$ at $x = 0$ b) $\mathbf{y} = (4\mathbf{x} - 5)^2$ at $\mathbf{x} = 0.5$

6. Determine the y coordinates and the gradient at the points on the following curves corresponding to the given values of x:

   a) $\mathbf{y} = \mathbf{x}^2 + \mathbf{x} + 1$ at $\mathbf{x} = 1,\ \mathbf{x} = 2$ b) $\mathbf{y} = \frac{2}{\mathbf{x}}$ at $\mathbf{x} = 2,\ \mathbf{x} = -4$

7. Differentiate each of the following with respect to x:

   a) $y = 3x + 2$
   b) $\mathbf{y} = \mathbf{x}^2 + 2\mathbf{x} - 1$
   c) $\mathbf{y} = (2 - 5\mathbf{x}^2)(\mathbf{x} + 1)$
   d) $\mathbf{y} = (\mathbf{x}^2 + 2)(\mathbf{x} - 1)^3$
   e) $y = 3x^4 + 2\sqrt{x}$
   f) $\mathbf{y} = (6\mathbf{x} + 2)(\mathbf{x} - 3)$

8. Sketch the curve and show intercepts, maxima and minima, and points of inflection of the following curves:

   a) $\mathbf{y} = \mathbf{x}^5 - 5\mathbf{x}^4$ b) $\mathbf{y} = \mathbf{x}^3 - 9\mathbf{x}$

9. A particle moves along the $\mathbf{x}-$axis. Its displacement at time t is given by $\mathbf{x(t)} = \mathbf{t}^3 + 2\mathbf{t}^2 - 4\mathbf{t} + 1$. Find its velocity and acceleration as the functions of time $\mathbf{t}$.

10. The sand is leaking through a small hole at the bottom of a conical funnel at the rate of 12 cm³/s. If the radius of the funnel is 8 cm and the altitude of the funnel is 16 cm. Find the rate at which the depth of the sand falls when the sand is 10 cm deep inside the cone.

11. A ladder 8 cm long is leaning against a vertical wall. If the top of the ladder is slipping down from the wall at 0.4 m/s, how fast does the bottom of the ladder slide along the ground when the bottom of the ladder is 3 cm from the wall?

12. A certain firm is facing the cost of production given by the function $\mathbf{C(x)} = 40 + 18\mathbf{x}^2 - 10\mathbf{x}$. Determine the
    a) marginal cost function
    b) total cost if $x = 2$
    c) maximum or minimum cost

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Chapter Five

Integration

Introduction

Integration or anti-derivative of a function is the reverse of differentiation. It is a process of finding a function whose derivative is known. The anti-derivative of the function represents a family of definite and indefinite integrals. Some real-life applications of integration involves finding displacement, velocity, and acceleration of moving bodies and determining the total cost and total revenue of produced goods. In this chapter, you will learn about the meaning of integration (also known as anti-derivative), distinguish integration from differentiation, determine definite and indefinite integrals, determine area under a given curve and area between two curves, find volumes of solid of revolution and apply integration in solving real- life problems.

5.1 The anti-derivative

Meaning of anti-derivative

Integration is the reverse of differentiation. In differentiation, if $f(x)=2x^2$, then $f'(x)=4x$. Thus, the anti-derivative of $4x$ is $2x^2$. That is, the process of reversing $f'(x)$ to $f(x)$ is called anti-derivative (integration). By similar reasoning, the integral of $2t$ is t^2. Integration can also be viewed as a process of summing or adding parts together. The symbol $\int$ is used to denote an integral, which means summation. Hence, referring the above functions, $\int 4x\,dx=2x^2$ and $\int 2t\,dt=t^2$. However, the same family of curves such as $y=2x^2, y=2x^2-5, y=2x^2+8$ and so on, have the same derivative $f'(x)=4x$, but when $4x$ is integrated, it is difficult to know which one among the family members was differentiated, thus the arbitrary constant "c" is usually added to generalise all the family members. Therefore $\int 4x\,dx=2x^2+c$ and $\int 2t\,dt=t^2+c$.

General formula for the anti-derivative

The general formula for anti-derivative can be deduced by considering the anti-derivatives of polynomials $x^0, x^1, x^2, x^3, \ldots, x^n$. Thus,

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$$\int x^0\,dx=\frac{x^{0+1}}{0+1}+c=x+c,\quad \int x^1\,dx=\frac{x^{1+1}}{1+1}+c=\frac{x^2}{2}+c$$

$$\int x^2\,dx=\frac{x^{2+1}}{2+1}+c=\frac{x^3}{3}+c,\quad \int x^3\,dx=\frac{x^{3+1}}{3+1}+c=\frac{x^4}{4}+c$$

Generally, $\int x^n\,dx=\frac{x^{n+1}}{n+1}+c;\ n\neq -1$, where c is an arbitrary constant.

For more examples, see Table 5.1 which shows the integration of some polynomial functions and their corresponding differentiation.

Table 5.1 Derivatives and Anti-derivatives of some functions

Function	Derivative	Anti-derivative (Integration)
$y=x^2$	$\frac{dy}{dx}=2x$	$\int 2x\,dx=x^2+c$
$y=x^4$	$\frac{dy}{dx}=4x^3$	$\int 4x^3\,dx=x^4+c$
$y=\frac{1}{5}x^5$	$\frac{dy}{dx}=x^4$	$\int x^4\,dx=\frac{1}{5}x^5+c$
$y=c$	$\frac{dy}{dx}=0$	$\int 0\,dx=c$
$y=\frac{1}{n+1}x^{n+1}+c$	$\frac{dy}{dx}=x^n$	$\int x^n\,dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$

5.2 Indefinite integral

Indefinite integral is an integral which does not have the limits at the end of the integral sign. The results of indefinite integral is always added with an arbitrary constant c.

Example 5.1 Find the following:

a) $\int x\,dx$ b) $\int x^2\,dx$ c) $\int t^3\,dt$ d) $\int 5\,dx$

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Solution

a) $\int x\,dx = \frac{1}{1+1}x^{1+1}+c = \frac{1}{2}x^2+c$

b) $\int x^2\,dx = \frac{1}{2+1}x^{2+1}+c = \frac{1}{3}x^3+c$

c) $\int t^3\,dt = \frac{1}{3+1}t^{3+1}+c = \frac{1}{4}t^4+c$

d) $\int 5dx = 5\frac{x^{0+1}}{0+1}x^{0+1} = 5x+c$

Example 5.2: Find the following:

a) $\int 4x^5\,dx$ b) $\int \frac{6}{x^3}dx$

Solution

a) $\int 4x^5\,dx = \frac{4x^6}{6}+c = \frac{2}{3}x^6+c$

b) $\int \frac{6}{x^3}dx = \int 6x^{dx} = \frac{6x^{-3+1}}{-3+1}+c = -3x^{-2}+c$

Exercise 5.1

Find the following:

1) $\int x^3\,dx$ 2) $\int 5x\,dx$

3) $\int \frac{1}{3x^2}dx$

4) $\int a^2x^7\,dx$ where $a \in \mathbb{R}$

5) $\int \sqrt{x}\,dx$ 6) $\int 5u^{\frac{3}{2}}du$

7) $\int \frac{2}{\sqrt[3]{t}}dt$ 8) $\int 5\sqrt[8]{x^3}\,dx$

9) $\int x^0\,dx$

10) $\int \frac{tdy}{\sqrt{y}}$, where $t \in \mathbb{R}$

The integral of a scalar multiple, sum and differences of functions

The integral of a function which is multiplied by a scalar quantity is equal to the integral of the function multiplied by the scalar quantity. Likewise, the integral of sum or differences of two or more functions is the sum of the individual integrals of difference between individual integrals. Symbolically this can be written as follows:

a) $\int af(x)dx = a\int f(x)dx$

b) $\int \big(f(x)\pm g(x)\big)dx = \int f(x)dx \pm \int g(x)dx$

where a is a constant.

Example 5.3: Find the following:

a) $\int (w^3+w^2+w)\,dw$

b) $\int \left(5-x^2+\frac{18}{x^4}\right)dx$

Solution

Each term is integrated separately as follows:

a) $\int (w^3+w^2+w)dw = \int w^3\,dw + \int w^2dw + \int w\,dw$

$= \frac{1}{3+1}w^{3+1} + \frac{1}{2+1}w^{2+1} + \frac{1}{1+1}w^{1+1}+c$

$= \frac{1}{4}w^4 + \frac{1}{3}w^3 + \frac{1}{2}w^2 + c$

Therefore,

$\int (w^3+w^2+w)dw = \frac{1}{4}w^4 + \frac{1}{3}w^3 + \frac{1}{2}w^2 + c$

b) $\int \left(5-x^2+\frac{18}{x^4}\right)dx = \int 5dx - \int x^2dx + \int \frac{18}{x^4}dx$

$= \int 5dx - \int x^2dx + 18\int x^{-4}dx$

$= 5x - \frac{1}{2+1}x^{2+1} + \frac{18}{-4+1}x^{-4+1} + c$

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$$= 5x - \frac{x^3}{3} - \frac{6}{x^3} + c$$

Therefore,

$$\int\left(5 - x^2 + \frac{18}{x^4}\right)dx = 5x - \frac{x^3}{3} - \frac{6}{x^3} + c$$

Example 5.4: Find the equation of the curve y for which $\frac{dy}{dx} = 2x + 1$ which passes through the point $(2,9)$.

Solution

Recalling that integrating a function is the reverse of differentiation; then, to find the equation of the curve for the given derivative use integration.

$$y = \int(2x+1)dx = x^2 + x + c$$

$$y = x^2 + x + c$$

In order to obtain the value of c, substitute $x = 2$ and $y = 9$

$$9 = 2^2 + 2 + c$$

$$9 = 4 + 2 + c$$

$$c = 3$$

$$\therefore \quad y = x^2 + x + 3$$

Exercise 5.2

Find the following:

1) $\int(5x-7)dx$ 2) $\int\left(\frac{8x^5 - 5x}{x^4}\right)dx$

3) $\int\sqrt[3]{4x}\,dx$ 4) $\int\frac{1}{2x^4}dx$

5) $\int\frac{(5t+4)(2t-3)}{\sqrt{t}}dt$

Integration by substitution

Some functions cannot be integrated using the basic formula for integrating polynomials. Instead, the technique of integration by substitution is used as it is shown in Example 5.5.

Example 5.5: Integrate the following with respect to x

a) $y = (3x+5)^4$ b) $y = x(x^2+1)^4$

c) $y = (9x-7)^{\frac{1}{2}}$ d) $y = \frac{3x+2}{(x^3+2x)^5}$

Solution

a) $y = \int(3x+5)^4\,dx$

Let $u = 3x+5$

$$du = 3dx \Rightarrow dx = \frac{du}{3}$$

$$\int(3x+5)^4\,dx = \int u^4\frac{du}{3}$$

$$= \frac{1}{3}\int u^4 du$$

$$= \frac{1}{3}\times\frac{1}{5}u^5 + c$$

But $u = 3x+5$, then:

$$\therefore \int(3x+5)^4\,dx = \frac{1}{15}(3x+5)^5 + c$$

b) $y = (9x-7)^{\frac{1}{2}}$

Let $u = 9x-7$, then

$$du = 9dx \Rightarrow dx = \frac{du}{9}$$

$$\therefore \int(9x-7)^{\frac{1}{2}}dx = \int(u)^{\frac{1}{2}}\frac{du}{9} = \frac{1}{9}\int(u)^{\frac{1}{2}}du$$

$$= \frac{1}{9}\frac{\left(u^{\frac{1}{2}}\right)}{\left(\frac{3}{2}\right)} + c. \text{ But } u = 9x-7$$

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$$\therefore \int (9x-7)^{\frac{1}{2}}\,dx = \frac{2}{27}(9x-7)^{\frac{3}{2}} + c$$

c) $y = x(x^2+1)^4$

Let $u = x^2+1$

Then $du = 2xdx \Rightarrow dx = \frac{du}{2x}$

$$\int x(x^2+1)^4 = \frac{1}{2}\int (u)^4\,du$$

$$= \frac{1}{2}\times\frac{1}{5}u^5 + c$$

$$\therefore \int x(x^2+1)^4\,dx = \frac{1}{10}\left(x^2+1\right)^5 + c$$

d) $\int \frac{3x^2+2}{(x^3+2x)^5}dx$

Let $u = x^3+2x$

Then, $du = (3x^2+2)dx \Rightarrow dx = \frac{du}{3x^2+2}$

Let $u = x^3+2x$

$$\therefore \int \frac{3x^2+2}{(x^3+2x)^5}dx = \int \frac{3x^2+2}{u^5}\cdot\frac{du}{3x^2+2}$$

$$= \int \frac{du}{u^5} = \int u^{-5}du$$

$$= \frac{1}{-5+1}u^{-5+1} + c = \frac{1}{-4}u^{-4} + c$$

$= \frac{-1}{4}\left(x^3+2\right)^{-4} + c$, since $u = x^3+2x$

$$\int \frac{3x^2+2}{(x^3+2x)^5}dx = \frac{-1}{4\left(x^3+2\right)^4} + c$$

Since there is an arbitrary constant c in each of these solutions, these are indefinite integrals.

Exercise 5.3

By using the substitution technique, integrate the following with respect to x:

1) $3x^3(x^4-7)^6$ 2) $x^2\sqrt{x^3+16}$

3) $x\ (x^2-1)^5$ 4) $x^6(x^7-4)^7$

5) $(3x^2+5)(x^3+3x-10)^4$

6) $\frac{(2x+3)}{4\sqrt{x^2+3x-2}}$

5.3 The definite integral

The result of finding an indefinite integral is usually a function with a constant of integration. In this part, the result of a definite integral is a definite number with no constant of integration. Definite integrals are used to find areas bounded by curves, volumes of solids, and many others.

Let $f(x)$ be a function. Then, the definite integral which sum the parts together between two points a and b is denoted by $\int_a^b f(x)\,dx$ and is read as the integral of the $f(x)$ from point a to point b. The points a and b are called the lower limit and upper limit, respectively. If

$\int f(x)\,dx = g(x)+c$,

Then $\int_a^b f(x)\,dx = g(x)+c\Big|_a^b$

$= \left(g(b)+c\right)-\left(g(a)+c\right)$

$= g(b)+c-g(a)-c$

$= g(b)-g(a)+c-c$

$= g(b)-g(a)+0$

$\therefore \int f(x)\,dx = \left[g(x)\right]_a^b = g(b)-g(a)$

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Example 5.6 Evaluate the following:

a) $\int_2^6 x\,dx$ b) $\int_{-1}^{2}(x^3+2x)\,dx$

Solution

a) $\int_2^6 x\,dx=\left[\frac{x^2}{2}\right]_2^6$

$=\frac{6^2}{2}-\frac{2^2}{2}$

$=\frac{36}{2}-\frac{4}{2}$

$=18-2$

$\therefore \int_2^6 x\,dx=16$

b) $\int_{-1}^{2}(x^3+2x)\,dx$

$\int_{-1}^{2}(x^3+2x)\,dx=\left[\frac{x^4}{4}+x^2+c\right]_{-1}^{2}$

$=\left(\frac{2^4}{4}+2^2\right)-\left(\frac{(-1)^4}{4}+(-1)^2\right)$

$=8-1\frac{1}{4}$

$\therefore \int_{-1}^{2}(x^3+2x)\,dx=\frac{27}{4}=6\frac{3}{4}$

Properties of definite integration

Evaluating integrals by applying the basic rule needs careful attention. Thus, it is important to use techniques for doing this efficiently. The properties of definite integrals are classified into two categories depending on arithmetic of integration and arithmetic for the domain (interval) of integration. The following are some important properties of integration.

a) Arithmetic properties of integration

i) $\int_a^b f(x)+g(x)\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx$

ii) $\int_a^b\big(f(x)-g(x)\big)dx=\int_a^b f(x)\,dx-\int_a^b g(x)\,dx$

iii) $\int_a^b cf(x)\,dx=c\int_a^b f(x)\,dx$

b) Interval properties of integration

i) $\int_a^b f(x)\,dx=0$

ii) $\int_b^a f(x)\,dx=\int_a^b f(x)\,dx$

iii) $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$

Exercise 5.4

Evaluate the following:

1) $\int_{-2}^{4} x\,dx$

2) $\int_2^2(x^2+x^3+x^4)\,dx$

3) $\int_{-2}^{3}3x^2\,dx$

4) $\int_1^4\left(\sqrt{x}+4x^4-\frac{5}{x^6}\right)dx$

5) $\int_{-2}^{3}x^5\,dx$

6) $\int_0^2(6x^2-4x+5)\,dx$

7) $\int_{-2}^{4}\sqrt[3]{5y^2}\,dy$

8) $\int_0^2(2x+7)^{12}dx$

9) $\int_1^2 x\sqrt{x^2+3}\,dx$

10) $\int_0^{2}\frac{1}{\sqrt{3x+1}}\,dx$

11) $\int_1^3(x^2+3x+1)\,dx$

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5.4 Applications of integration

Integration has many applications in science, economics, business and engineering. The focus of this topic is to use integration to find area under the curve, area between two curves and volume of solids of revolutions.

a) Area under the curve

One of the important applications of integration is to find the area bounded by a curve and a line. Often, such area can have a physical significance in describing the work done or the distance travelled. This part explains how such area is calculated.

i) The area of the curve above the x-axis

Consider the area bounded by the curve $y=f(x)$ between $x=a$ and $x=b$ and the x-axis as shown in Figure 5.1.

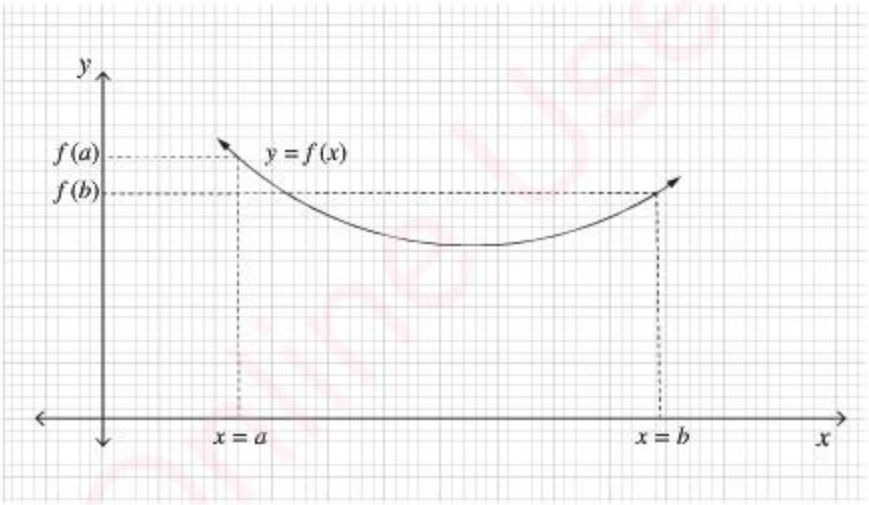

Figure 5.1 The graph of $y=f(x)$

The area under the curve $y=f(x)$ from $x=a$ and $x=b$ is computed using the idea of definite integral, that is

$$\int_a^b f(x)\,dx=\Big[F(x)\Big]_a^b=F(b)-F(a)$$

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Example 5.7: Find the area under the curve $f(x)=x^2+1$ from $x=0$ to $x=2$.

Solution

To find the area under the curve, first identify the required area by sketching the graph.

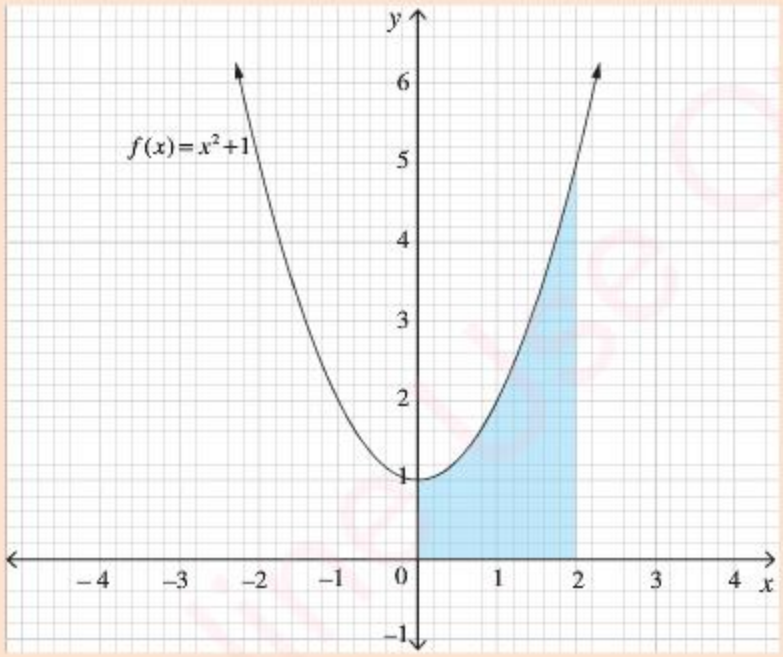

Figure 5.2 The graph of $f(x)=x^2+1$ from $x=0$ and $x=2$

The shaded region in the above sketch is the required area to be evaluated.

The area is:

$$\int_0^2 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_0^2$$

$$= \left(\frac{2^3}{3}+2\right)-\left(\frac{0^3}{3}+0\right)$$

$$= \left(\frac{8}{3}+2\right)-0 = \frac{14}{3}$$

$\therefore$ The area under the curve is $\frac{14}{3}$ square units

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Example 5.8: Find the area under the curve $f(x)=x^3+1$ from $x=-1$ to $=x=0$

Solution

Identify the required area by sketching the area of the curve $f(x)=x^3+1$ as shown in Figure 5.3.

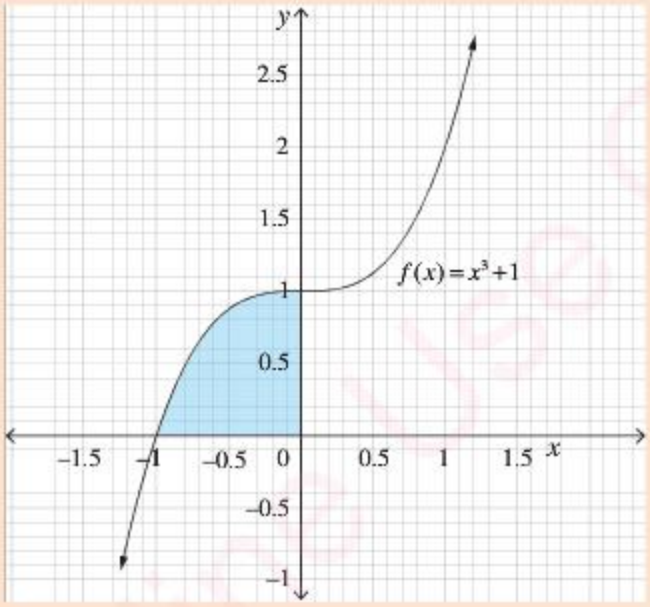

Figure 5.3 The graph of $f(x)=x^3+1$

$$\int_{-1}^{0}(x^3+1)\,dx=\left[\frac{x^4}{4}+x\right]_{-1}^{0}$$

$$=\left(\frac{0^4}{4}+0\right)-\left(\frac{(-1)^4}{4}+(-1)\right)$$

$$=0-\left(\frac{1}{4}+(-1)\right)$$

$$=-\frac{1}{4}+1$$

$\therefore$ The area under the curve is $\frac{3}{4}$ square units

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ii) The Area under the curve below the x – axis

Consider the area bounded by the curve $y = f(x)$ which is below the x – axis and between $x = a$, $x = b$ and the x – axis itself as shown in Figure 5.4.

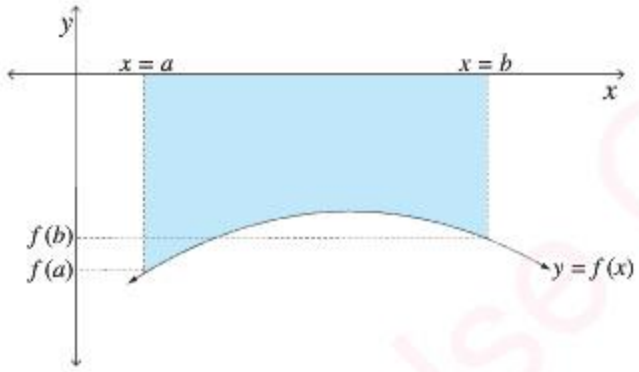

Figure 5.4 Area bounded by the curve and the x – axis below the x – axis

The resulting area in this case is a negative value. Since the area is always positive, the negative sign taken as an indication that the area lies below the x – axis but the value for the area is positive.

Example 5.9: Find the area under the curve $f(x) = -x^2 + 1$ from $x = 1$ to $x = 2$.

Solution: Consider the graph of $f(x) = -x^2 + 1$

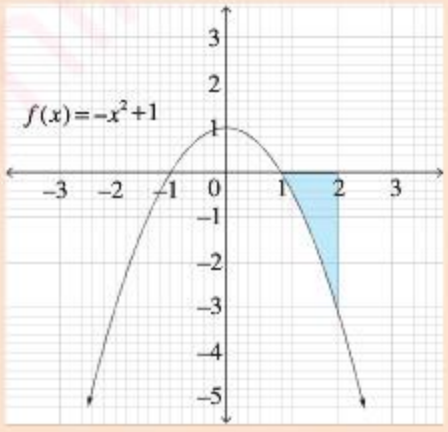

Figure 5.5 The graph of $f(x) = -x^2 + 1$

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$$\text{Area} = \int_1^2 (-x^2+1)\,dx = \left[-\frac{x^3}{3}+x\right]_1^2$$
$$= \left(\left[-\frac{8}{3}+2\right]\right) - \left(\left[-\frac{1}{3}+1\right]\right)$$
$$= -\frac{2}{3} - -\frac{2}{3}$$
$$= -\frac{4}{3}$$

Hence, the actual area is $\frac{4}{3}$ square units.

Remark: To avoid negative value of the area, the following formula may be used:

$$A = -\int_a^b f(x)\,dx \text{ or } A = \left|\int_a^b f(x)\,dx\right|$$

If the area under consideration contains parts above and below the $x-$axis, care must be taken when calculating this area. It is necessary to determine which parts of the graph lie above and below the $x-$axis. Separate integral values need to be calculated for each part of the graph. This idea is illustrated in Example 5.10.

Example 5.10: Find the total area enclosed by the curve $\mathbf{f(x)=4x^2-1}$ and the $x-$axis between $\mathbf{x=0}$ and $\mathbf{x=1}$

Solution

Required to determine which parts of the graph lie above and which one lie below the $x-$axis.

To do this, it is helpful to consider where the graph cuts the $x-$axis. So consider the function $\mathbf{f(x)=4x^2-1}$ and find its zeros.

The graph of the function is illustrated in figure 5.6

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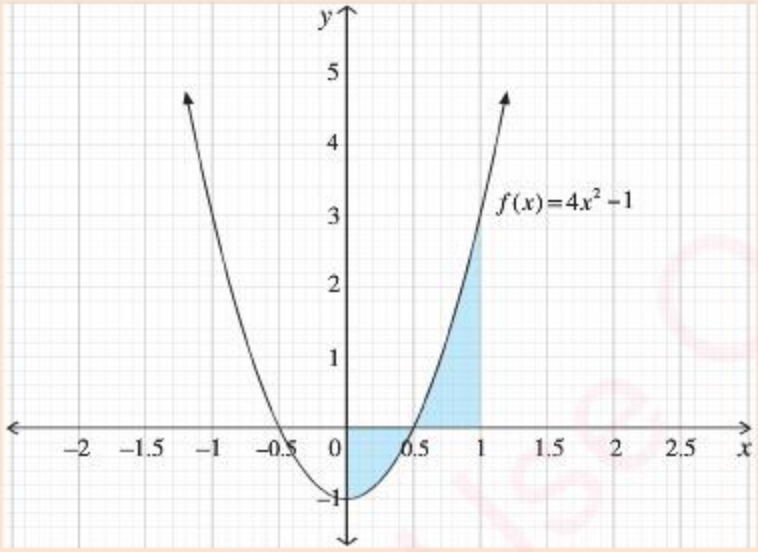

Figure 5.6 The graph of $f(x)=4x^2-1$

The graph cuts the $x-$axis when $x=-0.5$ and $x=0.5$. From the graph, the required area lies partly below the $x-$axis when $0\le x\le 0.5$ and partly above the $x-$axis when $0.5\le x\le 1$.

So the integral is evaluated in two parts:

$$\int_0^1\left(4x^2-1\right)dx=\int_0^{0.5}\left(4x^2-1\right)dx+\int_{0.5}^1\left(4x^2-1\right)dx$$

$$=\left|\left[\frac{4x^3}{3}-x\right]_0^{0.5}\right|+\left|\left[\frac{4x^3}{3}-x\right]_{0.5}^1\right|$$

$$=\left|\left(\frac{4(0.5)^3}{3}-0.5\right)-\left(\frac{4(0)^3}{3}-0\right)\right|+\left|\left(\frac{4(1)^3}{3}-1\right)-\left(\frac{4(0.5)^3}{3}-0.5\right)\right|$$

$$=\left|\left(\frac{4\times 0.125}{3}-0.5\right)-0\right|+\left|\left(\frac{4}{3}-1\right)-\left(\frac{4\times 0.125}{3}-0.5\right)\right|$$

$$=\left|\left[\left(\frac{0.5}{3}-0.5\right)-0\right]\right|+=\left|\left[\left(\frac{4}{3}-1\right)-\left(\frac{0.5}{3}-0.5\right)\right]\right|$$

$$=\left|-\frac{1}{3}\right|+\left|\frac{2}{3}\right|=1$$

Therefore, the required area of the curve is 1 square unit

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Exercise 5.5

1. Find the area under the curve $y = x^2 + 3$ from $x = 0$ to $x = 2$.
2. Sketch the curve $y = x^2 - 3x + 2$, and find the area between the curve and the $x-$axis form $x = 0$ to $x = 3$.
3. Find the area bounded by the function $y = x^2 + 3$ from $x = 0$ to $x = 2$ and the $x-$axis.
4. Find the area under the curve $y = x^2 - 5x + 4$ and the $x-$axis.
5. Find the enclosed area between the arc $y = x^2(x-1)(3-x)$ and the $x-$axis between the lines $x = 1$ and $x = 3$.

iii) Area between two intersecting curves

If two curves $y = f(x)$ and $y = g(x)$ intersect at two points A and B as shown in Figure 5.7, the area between these two curves can be determined.

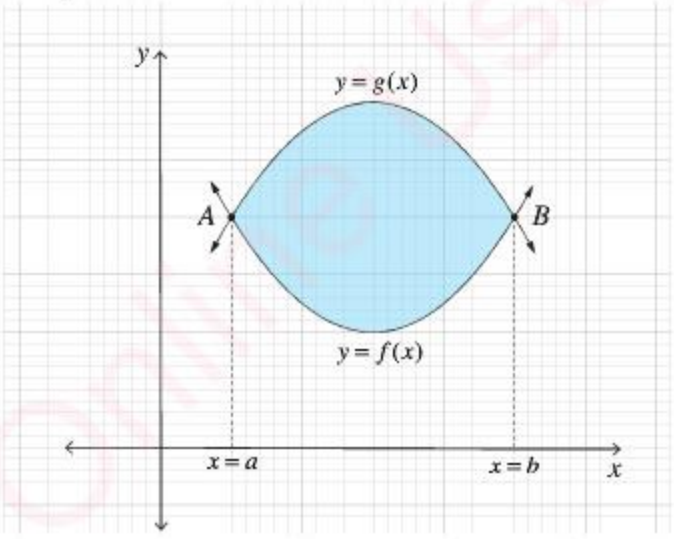

Figure 5.7 Area between intersecting curves.

The area between the two curves can be found as follows.

Area A_1 under the curve $y = g(x)$ between $x = a$ and $x = b$ is given by $A_1 = \int_a^b g(x)\,dx$. Likewise, area A_2 under the curve $y = f(x)$ between $x = a$ and $x = b$ is given by $A_2 = \int_a^b f(x)\,dx$. The area A between two curves is therefore, the difference between areas A_1 and A_2. That is, $A = A_1 - A_2$

$$A = \int_a^b g(x)dx - \int_a^b f(x)dx$$

$$\therefore A = \int_a^b (g(x) - f(x))dx$$ where $|g(x)| > |f(x)|$ in the interval $a \le x \le b$

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Remark: In this case, it does not matter whether the shaded area is above or below or cut into parts by the $x-$axis.

Example 5.11: Find the area bounded by the curve $f(x)=x^2$ and the straight line $g(x)=x$.

Solution

The graph of the curve $f(x)=x^2$ and the straight line $g(x)=x$ is shown in Figure 5.8.

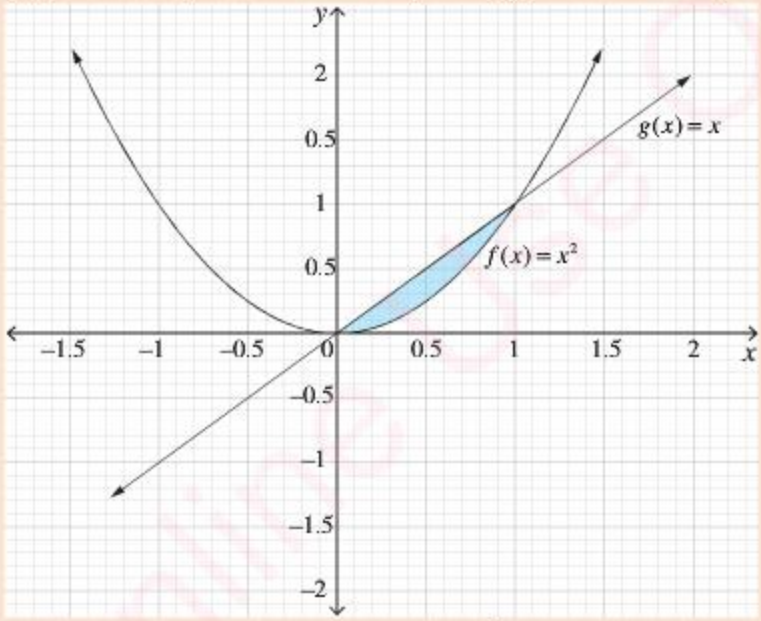

Figure 5.8 The graph of intersecting curve $f(x)=x^2$ and the line $g(x)=x$

Find the values of x where the curves and the straight line meet by equating the two functions. Therefore,

$$f(x)=g(x)$$

$$x^2=x$$

$$x^2-x=0$$

$$x(x-1)=0$$

$$x=0 \text{ or } x=1$$

The lower curve is subtracted from the upper curve in order to evaluate the area of the shaded region as follows: Area

$$=\int_a^b[f(x)-g(x)dx]$$

$$=\int_0^1(x-x^2)dx=\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$$

$$=\frac{1}{2}-\frac{1}{3}$$

Therefore, area is $\frac{1}{6}$ square units.

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Example 5.12: Find the area bounded by the curves $f(x)=x^2$ and $g(x)=2x-x^2$

Solution

First step: sketch the curves as shown below:

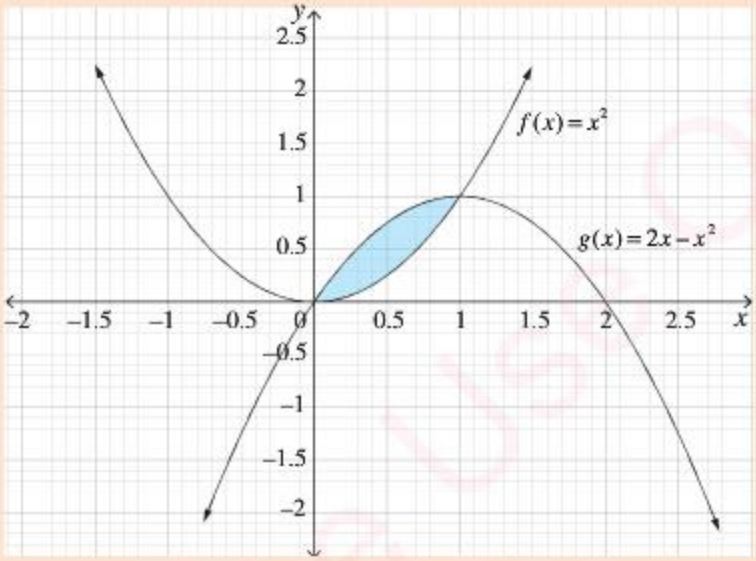

Figure 5.9 The graph of $f(x)=x^2$ and $g(x)=2x-x^2$

Second step: find where the curves meet, that is, where $f(x)=g(x)$ to obtain the limits of integration.

$$f(x)=g(x)$$
$$x^2=2x-x^2$$
$$2x^2-2x=0$$
$$2x(x-1)=0$$
$$x=0 \text{ or } x=1$$

So $x=0$ or $x=1$ are the limits of integration.

Using $A=\int_0^1(g(x)-f(x))\,dx$,

$$A=\int_0^1((2x-x^2)-x^2)dx=\int_0^1(2x-2x^2)dx$$
$$=\left[x^2-\frac{2}{3}x^3\right]_0^1$$
$$=1-\frac{2}{3}=\frac{1}{3}$$

Therefore, the area bounded by the two curves is $\frac{1}{3}$ square units.

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Exercise 5.6

1. Find the area enclosed by the curves $f(x)=2x^2+7x+3$ and $g(x)=9+4x-x^2$.
2. Find the area enclosed by the curves $y=x^2$ and $y=2x^2-4$.
3. Find the area enclosed by the curves $y=x^2$ and $y=\sqrt{x}$.
4. Find the area enclosed by the curve $y=x^2-4$ and the straight line $y=-2x$.
5. Find the area enclosed by the curve $y=x^3$ and the straight line $y=3x$.

b) Volume of solids of revolution

A solid of revolution is obtained by rotating the region bounded by the curve about the axes x or y. Figure 5.10 shows the curve $y=f(x)$ rotated about the $x-$axis to form a solid.

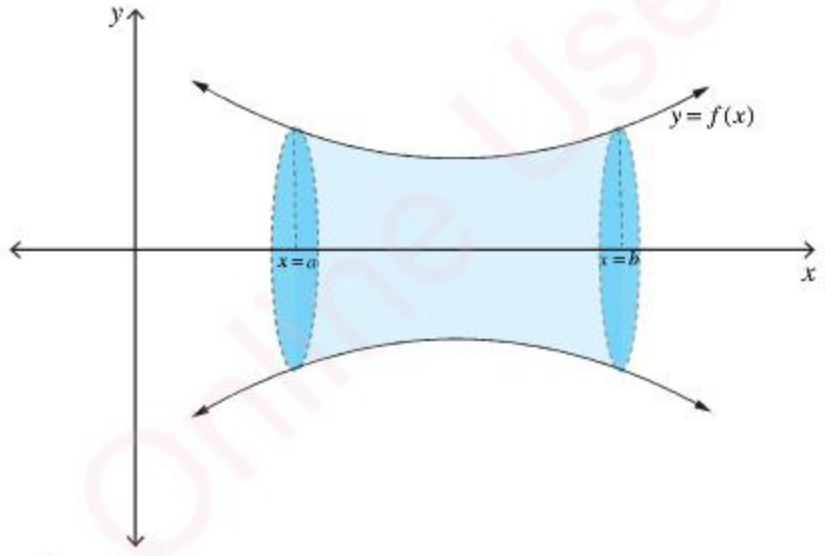

Figure 5.10 Solid of revolution formed by rotation

The volume of solid of the revolution obtained by revolving the curve $y=f(x)$ from $x=a$ to $x=b$ about the $x-$axis is given by;

$$V=\int_a^b \pi y^2 dx$$

Similarly, when the region is rotated about the y-axis from $y=a$ to $y=b$ the volume V is computed by the integral;

$$V=\int_a^b \pi x^2 dy$$

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Example 5.13: Find the volume of the solid formed by rotating the region bounded by a straight line $y = x$, the $x-$axis, and the lines $x = 1$ and $x = 3$ about the $x-$axis.

Solution

A sketch of the solid of revolution:

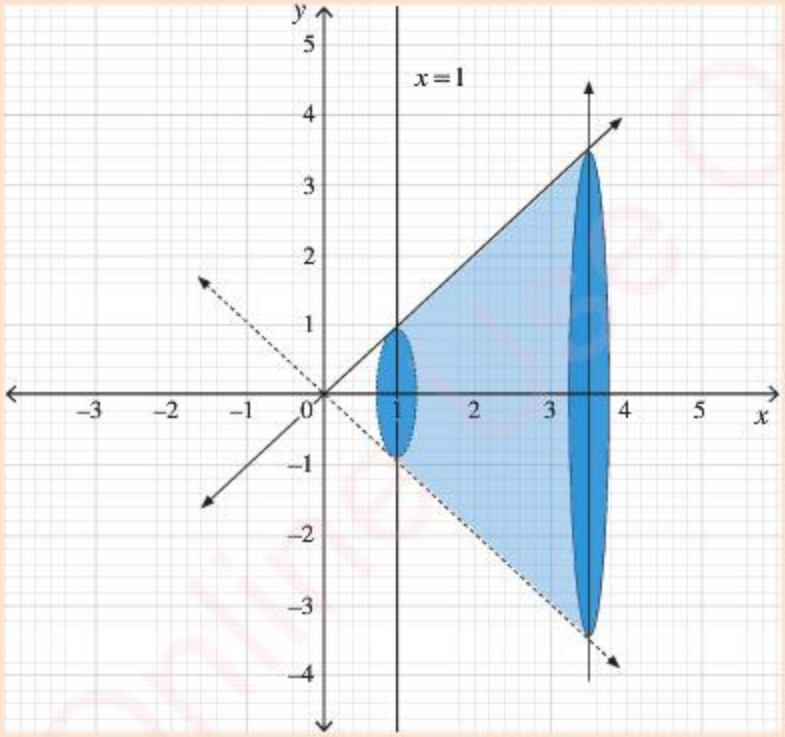

Figure 5.11 A sketch of the solid of revolution formed by a line $y = x$, the $x-$axis and the lines $x = 1$ and $x = 3$ about the $x-$axis

Using; Volume $= \int_a^b \pi y^2 dx$ and $y = x$

$$= \int_a^b \pi x^2 dx$$

$$= \left[\frac{\pi x^3}{3}\right]_1^3$$

$$= \pi\left(\frac{(3)^3}{3} - \frac{(1)^3}{3}\right)$$

$$= \pi\left(\frac{27}{3} - \frac{1}{3}\right)$$

Therefore the volume is $\frac{26\pi}{3}$ cubic units.

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Example 5.14: Find the volume of solid of revolution generated when the region between the curve $y=4-x^2$ and the line $y=0$ and $y=4$ is rotated about the $y-$axis.

Solution

Since the solid is rotated about the $y-$axis, the function $y=4-x^2$ should be written as $x^2=4-y$.

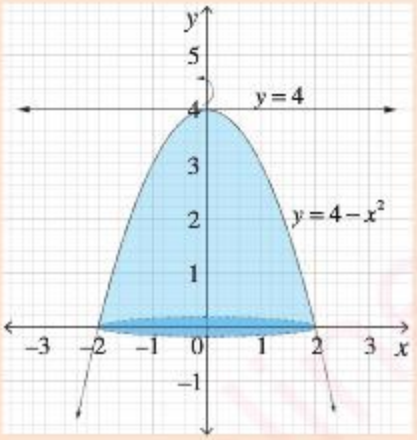

Figure 5.12 The graph of the volume of solid of revolution formed by the curve $y=4-x^2$ and the line $y=0$, $y=4$ rotated about the $y-$axis

The volume generated is calculated using $y=4$,

$$\text{Volume} = \int_a^b \pi x^2\,dy$$

$$= \int_0^4 \pi(4-y)\,dy$$

$$= \left[\pi\left(4y-\frac{y^2}{2}\right)\right]_0^4$$

$$= \pi\left(16-\frac{16}{2}\right)$$

$$= \pi(16-8) = 8\pi$$

$\therefore$ The volume of the solid of revolution is 8π cubic units

Example 5.15: Find the volume of solid generated by the curve $y=x^2$ rotating about the $-$axis, and the straight lines $=-4$ and $=-2$.

Solution

$$= \int \pi\ ^2$$

$$= \pi \int_{-4}^{-2}\ ^4$$

$$= \left[\frac{\pi\ ^5}{5}\right]_{-4}^{-2} = \frac{992\pi}{5} \text{ cubic units}$$

$\therefore$ The volume of the solid generated is $\frac{992}{5}\pi$ cubic units

Exercise 5.7

1. Find the volume formed by rotating the region between the curve $y=\sqrt{x+3}$ and the $-$axis from $x=0$ to $x=4$ about the $-$axis.
2. What is the volume generated when the area enclosed by the straight line $y=x$, $y-$axis, and the straight line $y=2$ is rotated about the $y-$axis?
3. Find the volume of the solid generated by revolving the region bounded by $y=x^2+2$ and $y=x+4$ about the $-$axis.

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4. Find the volume of the solid generated by revolving the region bounded by $y=x^2$ and the $x-$axis on $-2\le x\le 3$ about the $x-$axis.

5. Find the volume of the solid generated by revolving the region bounded by $y=x^2-1$ and the $y-$axis on $1\le y\le 9$ about the $y-$axis.

6. Find the volume of the solid generated by rotating the region bounded by $y=x^2$, $y-$axis and the straight line $y=2$ about the $y-$axis.

7. Find the volume of the solid of revolution generated by revolving about the $x-$axis and the region under the curve $y=\sqrt{r^2-x^2}$ from $x=-r$ to $x=r$.

Total cost and total revenue

The additional cost resulting from producing one unit of an output is known as the Marginal Cost while the increase in the total revenue generated when one unit of an output is produced is called the Marginal revenue. Now, assume x be the number of items produced or sold, $\mathrm{MC}(x)$ be the Marginal Cost, and $\mathrm{MR}(x)$ be the Marginal Revenue. The anti-derivative of the Marginal Cost and Marginal Revenue functions give the total cost ($\mathrm{C}(x)$) and total revenue ($\mathrm{R}(x)$) functions, respectively.

That is;

$R(x)=\int \mathrm{MR}(x)\,dx$ and

$\mathrm{C}(x)=\int \mathrm{MC}(x)\,dx$

Example 5.16: The Marginal Cost of the firm is given by $\mathrm{M}(x)=20+0.5x^2-3x^3$. Find the total cost function.

Solution

Given: $\mathrm{M}(x)=20+0.5x^2-3x^3$

Using; $\mathrm{C}(x)=\int \mathrm{MC}(x)\,dx$

$$=\int (20+\frac{1}{2}x^2-3x^3)dx$$

$$=20x+\frac{1}{6}x^3-\frac{3}{14}x^4+c$$

$$\therefore \mathrm{C}(x)=20x+\frac{1}{6}x^3-\frac{3}{14}x^4+c$$

Example 5.17: Given that the fixed cost is 30 and the Marginal Cost, $\mathrm{MC}(x)=3x-\frac{1}{2}x^2$. Determine the total Cost function.

Solution

Using; $\mathrm{C}(x)=\int \mathrm{MC}(x)\,dx$ and $\mathrm{FC}=30$

$$=\int\left(3x-\frac{1}{2}x^2\right)dx$$

$$=\frac{3}{2}x^2-\frac{1}{6}x^3+\mathrm{c}$$

Since $\mathrm{C}(x)=\mathrm{FC}+\mathrm{VC}(x)$ where FC represents Fixed Cost VC represents Variable Cost and $C(x)$ represents Total Cost.

Therefore, $\mathrm{C}(x)=30+\frac{3}{2}x^2-\frac{1}{6}x^3$

meaning that FC = C = 30

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Example 5.18: Assume that the marginal cost in Tanzanian shillings is given by $\mathrm{MC}(x)=\frac{3}{7}e^{-2x}+6x^2+11$, where x is the quantity produced. Find the total cost of production when $x=3$, if the fixed cost is Tsh 10,000.

Solution:

Since the marginal cost is given by $\mathrm{MC}(x)=\frac{3}{7}e^{-2x}+6x^2+11$, then the total cost

$$\mathrm{C}(x)=\int \mathrm{MC}(x)\,dx$$

$$\mathrm{C}(x)=\int\left(\frac{3}{7}e^{-2x}+6x^2+11\right)dx$$

Thus, $\mathrm{C}(x)=-\frac{3}{14}e^{-2x}+2x^3+11x+T$

where T is the arbitrary constant of integration. Now, $\mathrm{C}(x)=10000$ when $x=0$,

$$10000=-\frac{3}{14}+T \qquad T=10000.21$$

Hence,

$$C(3)=-\frac{3}{14}e^{-2(3)}+2(3)^3+11(3)+10000.21$$
$$=1087.2095$$

Therefore, the total cost of production is Tsh 1,087.2095.

Exercise 5.8

1. Given that the Marginal Cost of a certain commodity is equal to half of its average cost.
   (a) Show that the fixed cost is zero.
   (b) If the cost of producing 7 units of the commodity is Tsh 900, find the cost function.

   Hint: Average cost$=\frac{\mathrm{C}(x)}{x}$

2. The Marginal Cost of a product is given by $2-3x+x^2$ where x is the number of units of a product. If x increases from 50 to 75 units, find the total increase in cost.
3. The total cost of producing a pair of suit is Tsh 50,000. If its marginal cost function of manufacturing x suits is $3000-50x+5x^2$, find the total and average cost functions when the number of suits produced is 2.
4. If the Marginal Revenue function of a certain firm is given by $MR(x)=30+2x$. Find the Total Revenue function if $MR(0)=16$.
5. A shop manufacturer Marginal Revenue function is $MR(x)=375-x-0.5x^2$. Find the increase in the manufacturer's Total Revenue if the production is increased from 20 to 30 units.

Displacement and velocity

Integration plays a major role in describing the motion of a body using vector quantities like displacement, velocity and acceleration; ignoring the factors causing the body's motion.

Determining the quantities of motion (displacement, velocity and acceleration) with respect to time can be done by differentiation and integration. For instance, the derivative of displacement with time is velocity, the derivative of velocity with time is acceleration; where as the integral of acceleration over time is velocity, and the integral of velocity over time is displacement.

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The integral of acceleration of the object with respect to time gives its velocity $V(x)=\int_{t_1}^{t_2} a(t)\,dt$. Likewise, the integral of velocity with respect to time gives the displacement $S(t)=\int_{t_1}^{t_2} v(t)\,dt$..

Example 5.19: A particle is moving along a straight line with a velocity of $V(t)=t^3-4t^2-4t+16$ metres per second at time t. What is the displacement of the particle within the time interval $1\le t\le 5$?

Solution

Displacement; $S(t)=\int_{t_1}^{t_2} V(t)\,dt$.

$$=\int_1^5 (t^3-4t^2-4t+16)dt$$

$$=\left[\frac{1}{4}t^4-\frac{4t^3}{3}-2t^2+16t\right]_1^5$$

$$=\left(\frac{(5)^4}{4}-\frac{4(5)^3}{3}-2(5)^2+16(5)\right)-\left(\frac{(1)^4}{4}-\frac{4(1)^3}{3}-2(1)^2+16(1)\right)$$

$$=\left(\frac{625}{4}-\frac{500}{3}-50+80\right)-\left(\frac{1}{4}-\frac{4}{3}-2+16\right)$$

$\therefore$ Displacement is $\frac{20}{3}$ metres

Example 5.20: A car starts from rest and moves a displacement (in metres) of $s(t)=\frac{1}{6}t^4+\frac{1}{4}t^3$. What is the velocity at the end of 2 seconds?

Solution

Since, $s(t)=\frac{1}{6}t^4+\frac{1}{4}t^3$

Then $\frac{d}{dt}(s(t))=\frac{d}{dt}\left(\frac{1}{6}t^4+\frac{1}{4}t^3\right)$

$$v(t)=\frac{4}{6}t^3+\frac{3}{4}t^2=\frac{2}{3}t^3+\frac{3}{4}t^2$$

$$v(2)=\frac{2}{3}(2)^3+\frac{3}{4}(2)^2=\frac{16}{3}+3$$

$\therefore$ $v(2)=8\frac{1}{3}$ m/s

Therefore, the velocity after 2 seconds is $8\frac{1}{3}$ m/s

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Exercise 5.9

1. A particle moves along a straight line so that its velocity at time t is $V(t)=(t^2-t-6)$m/s. Find the displacement of the particle during the time period $1\le t\le 4$.
2. The velocity of the car is given by $V(t)=3t^2-12$. Find the displacement travelled on the time interval $1\le t\le 3$.
3. Consider the particle which is moving in a vertical line with the velocity $19.6-9.8t$ measured in metre per second and time (t) in seconds. Find the displacement covered by the particle on the interval $t=1$ and $t=4$.
4. A taxi runs non-stop between two towns N and M and its velocity after leaving town N is $30t-15t^2$ km / hr. Find the displacement between the two towns if it takes 2 hours to reach town M.
5. The motion of a particle is given by $V(t)=1+5t+6t^2$. Find the displacement of a particle between $t=1$ and $t=3$.
6. The acceleration of a car is given by $(\ )=3\ ^2+\frac{3}{2}$ in m / s² calculate its velocity at $t=5$ seconds given that $(2)=13$ m / s.
7. A stone is thrown vertically downwards from the top of the cliff with a velocity given by $(\ )=2\ +18\ ^2$ in m / s in metres. Find displacement function given $(0)=0$.

Revision exercise 5

1. In problems a) – e), find the following:

   a) $\int\ ^5$ b) $\int\ ^{\frac{3}{4}}$ c) $\int\frac{1}{2}$ d) $\int\left(3\sqrt{\ }-\frac{2}{3}+\frac{1}{2}\right)$

   e) $\int\left(\ ^{\frac{1}{2}}-3\ ^{\frac{2}{3}}+6\right)$

2. Find the function whose slope is $+1$ and its graph passes through the point $(2,3)$.
3. Find the Cost function if the Marginal cost is $3+40\ -5\ ^2$ and the Fixed cost is 45.
4. Suppose is a function such that $'(\ )=\ ^2+1$ and $(3)=2$. Find $(\ )$.
5. Find the area of the region bounded by

   a) $=\sqrt{\ }$ and $=$ b) $=\ ^2-4\ +4$ and $=10-\ ^2$

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6. Find each of the following:

a) $\int (x-4)\,dx$ b) $\int x\sqrt{3x^2+7}\,dx$ c) $\int (2x+1)^8 dx$

d) $\int (7x^3-4x^2+2x-1)\,dx$ e) $\int\left(\frac{1}{2}x^2+4x-2\right)dx$ f) $\int x^2\left(x^3-4\right)^6\,dx$

7. The Marginal cost $MC(x)$ of a company is $1.5x^2-1.4x-45$. Determine the total cost incurred by the company.

8. Find the following:

a) $\int\left(3t^2+2\right)\left(t^3+2t-10\right)^3\,dt$ b) $\left(4t-1\right)^5\,dt$ c) $\int t^3\left(t^4-3\right)^7 dt$

9. Evaluate:

a) $\int_0^2 x^5\left(x^6-3\right)^4\,dx$ b) $\int_0^1 \frac{2x+3}{\sqrt{x^2+3x+1}}dx$

10. Evaluate:

a) $\int_2^4\left(x^2+2x\right)dx$ b) $\int_1^4\sqrt{x}\,dx$

11. a) Find the area enclosed by the curve $y=x^2-3x+2$ and the $x-$axis.
b) Determine the area of the region enclosed by the curves $y=x^2$ and $y=8-x^2$.

12. Find the value of n such that $\int_1^2 nx\left(2-x^2\right)^2 dx=1$.

13. Find the volume of the solid generated by rotating about $x-$axis the area bounded by $y^2=x^2+2$ and the $x-$axis between $x=1$ and $x=4$.

14. The Marginal Revenue function of a certain company was found to be

$\mathrm{MR}(x)=5+12x+16x^2$. Determine its Revenue function.

15. A particle moves along the straight line with the velocity V where $V(t)=\frac{t^4}{4}-t^3+t^2$. Find its displacement at the time interval between $0\le t\le 2$.

16. A point moves along a straight line so that its velocity (v) in meter per second at a time t is given by $v=27t-t^2$. Find its displacement between $t=3$ seconds and $t=4$ seconds.

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Chapter Six

Statistics

Introduction

Statistics is concerned with scientific methods of collecting, organizing, summarizing, presenting, and analysing data as well as deriving valid conclusions. It can be classified into two types; descriptive and inferential statistics. The knowledge of statistics is applied in different fields such as in education, business, medicine, and many others to inform stakeholders with key information for decision making. In this chapter you will learn and practice how to collect, organise, and present data in tables and graphs, calculate the measures of the central tendency, and apply computer packages in analysing statistical data.

6.1 Collection, organisation and presentation of data

Data collection

In statistics, collected information is known as data (in singular, datum). Methods used for data collection include: interviews, surveys, questionnaires and observation. The collected data which are not yet organized or arranged in a particular order are called raw data. It is therefore very difficult to assign meaning to the raw data or even draw conclusion out of it unless they are organized in one way or another.

The following are some common terms used in statistics.

a) **Data:** are information especially facts or numbers collected to be examined and considered to be used in decision making.
The singular of data is datum, this is when the observation has only a single outcome.

Types of data

i) **Quantitative data:** are numeric values which represents a measurable quantity.

ii) **Qualitative data**: are all non-numeric data i.e. data that have not been quantified. For example, gender i.e. male or female, taste of food i.e. sweet or bitter, good, very good and excellent.

b) **Variable:** Is a characteristic, number or quantity that increases or decreases over time. It is a property of an object or event that takes on different values in different situations.

c) **Population:** Is a statistical data that contain all members of a specified group. It is the field from which data or sample can be extracted.

d) **Sample**: Is a set of data collected or selected from a statistical population by a defined procedure.

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Activity 6.1

Prepare a questionnaire which is to be used to collect data from any source nearby or within the school. Use the questionnaire to collect data from selected sources and summarize the data collected and present the findings in the class for discussion.

The frequency distribution table

The collected unorganized data are called raw data. Data should be arranged and presented in a concise way in order to observe the data trends and patterns easily. The process of arranging and presenting data in the table is called tabulation. The table prepared through the tabulation process is called frequency distribution table. In frequency distribution table, data are prepared by using tallies and the number of data entries in a particular class is called frequency denoted by the letter 'f'.

Data can be prepared in two forms which are:

a) Ungrouped data; where data are arranged as single entity.

b) Grouped data; where data are arranged by grouping into classes of specified interval sizes

Tallying procedures

Step 1: List the values or data in a column in ascending order by writing all possible values without repetition.

Step 2: Stating with the first row or column of the data, and moving from left to right or up to down, mark a sign against a value in the list as it occurs until all the values in the row or column are exhausted.

Step 3: Repeat Step 2 in the rest of rows or column until all the data are exhausted (confirm the total of all numbers with tallied numbers/data).

Example 6.1 Make a frequency distribution table for the following data:

5, 10, 7, 19, 25, 12, 15, 7, 6, 8, 17, 17, 22, 21, 7, 7, 24, 5, 6, 5

a) Without grouping.

b) By grouping into five classes.

Solution

a) Frequency distribution table for ungrouped data

Table 6.1 A frequency distribution table for ungrouped data

Data value	Tallies	Frequencies
5	///	3
6	//	2
7	////	4
8	/	1
10	/	1
12	/	1
15	/	1
17	//	2
19	/	1
21	/	1
22	/	1
24	/	1
25	/	1
		N=20

Where N represent the total number of frequencies.

b) Frequency distribution table for grouped data.

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Use the formula below for the class width

i.e class width (size) $= \dfrac{\text{Range}}{\text{Number of classes}}$

$= \dfrac{\text{Highest value} - \text{Lowest value}}{\text{Number of classes}}$

$= \dfrac{25-5}{5} = \dfrac{20}{5} = 4$

Note that the value should be rounded up to the nearest integer. For example $4.1 \approx 4$ and $4.9 \approx 5$.

Thus the first class interval can be obtained as follows:

$5-(5+4) \Rightarrow (5-9)$, by using the same method the second class interval can be obtained as follows $10-(10+4) \Rightarrow (10-14)$ etc.

Hence, a frequency distribution table is as presented in Table 6.2.

Table 6.2 A frequency distribution table for grouped data.

Class intervals	Tallies	Frequencies
5 – 9	//// ////	10
10 – 14	//	2
15 – 19	////	4
20 – 24	///	3
25 – 29	/	1
		N=20

Exercise 6.1

1. Prepare the frequency distribution table for the following ungrouped data without grouping:

   2, 1, 3, 4, 2,5, 2, 3, 4, 5, 1, 6, 2, 3, 2, 2, 3, 4, 4, 1, 3, 3, 2, 4, 5, 2, 6, 1, 4, 2, 3, 4

2. The following scores out of 25 were attained by 30 students who sat for the Biology test in a certain school:

   20, 6, 23, 19, 9, 14, 15, 3, 1, 12, 10, 20, 13, 3, 17, 10, 11, 6, 21, 9, 6, 10, 4, 5, 1, 5, 1, 5, 11, 7, 24

   Represent the data in a frequency distribution table by grouping the data into 5 classes.

3. Prepare the frequency distribution table for the following ungrouped data.

   43, 75, 50, 52, 51, 47, 50, 47, 40, 48

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4. Prepare the frequency distribution table for the following data set without grouping:
   8, 8, 7, 7, 7, 7, 1 , 9, 1, 1, 9, 1, 1, 5, 5, 2, 5, 6, 4, 6, 1, 5, 6.
5. The test scores for a class of 20 students are as follows:
   93, 84, 97, 100, 78, 86, 100, 85, 92, 72, 55, 91, 90, 75, 94, 83, 60, 81, 95.
   Use these data to complete Table 6.3.

Table 6.3 The frequency distribution table for the test scored by 20 students.

Class intervals	Class marks	Frequencies
91 – 100		
81 – 90		
71 – 80		
61 – 70		
51 – 60		

Cumulative frequency distribution table

Cumulative frequency of a set of data refers to the sum of the frequencies of the data up to a required defined level. The distribution table which comprises the data values, frequencies, and cumulative frequencies is called cumulative frequency distribution table.

Example 6.2 The set of data in the Table 6.4 shows the number of books possessed by students. Prepare a cumulative frequency table for the data:

Table 6.4 Number of books possessed by students

Books	Frequencies
1	1
2	3
3	2
4	4
5	8
6	2

Solution

Table 6.5 A cumulative frequency table

Books	Frequencies	Cumulative frequencies
1	1	1
2	3	1+3=4
3	2	4+2=6
4	4	6+4=10
5	8	10+8=18
6	2	18+2=20
	N = 20	

Some other useful terminologies in statistics

Class limits

There are two limits for each class interval in the grouped data, the lower limit and the upper limit. The lower limit for every class interval is the smallest value in that class and the upper limit for every class interval is the greatest value in that class.

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Class mark

This is the average of class limits of each class i.e.

$$\text{Class Mark} = \frac{\text{Lower limit} + \text{Upper limit}}{2}$$

Class boundary

Class boundaries are the numbers used to separate classes. The size of the gap between classes is the difference between the upper class limit of one class interval and the lower class limit of the next class interval. The lower boundary of each class is calculated by subtracting half of size of the gap between the classes from each lower limit class, while upper boundary of each class is calculated by adding half of the size of the gap between the classes to each upper limit class.

Class size

The difference between consecutive lower limits or upper limits is called a class size. Also a class size is the difference between the class boundaries of each class interval. But also it refers to the difference between consecutive class marks. A class size is sometime called class width. Table 6.7 in Example 6.3 summarises the values of class marks, class limits, class boundaries and class size.

Remark: When the class intervals have different sizes, the class size of each one is the difference between its class boundaries.

Example 6.3

Consider the information given in Table 6.6:

Table 6.6 Table for statistics terms

Class intervals	Frequency	Class Marks	Lower limits	Upper limits	Class boundaries	Lower boundaries	Upper boundaries
6 – 10	4						
11 – 15	5						
16 – 20	9						
21 – 25	2						
26 – 30	6						

By using information in Table 6.6

a) Complete the given table b) Find the class size

Solution

a) Completing the table

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Table 6.7 Table for solution of problem in Table 6.6

Class intervals	Frequencies	Class Marks	Lower limits	Upper limits	Class boundaries	Lower boundaries	Upper boundaries
6 – 10	4	8	6	10	1	5.5	10.5
11 – 15	5	13	11	15	1	10.5	15.5
16 – 20	9	18	16	20	1	15.5	20.5
21 – 25	2	23	21	25	1	20.5	25.5
26 – 30	6	28	26	30	1	25.5	30.5

b) Class size (width) = differences between consecutive lower limits or upper limits
i.e. $11-6=5$ or $15-10=5$

$\therefore$ The class size $=5$

Exercise 6.2

1. The following data show the number of Civics books that were surveyed in a sample of 50 schools at Lindi district.

42	24	11	05	47	27	13	27	28	33
45	30	25	31	25	32	03	15	07	50
19	21	20	32	28	22	35	16	38	14
39	20	11	58	14	37	26	21	35	20
42	07	27	32	24	22	31	28	40	36

Construct a frequency distribution table to represent the given data using the intervals 1 –9, 10 – 18, …

2. Two dice were rolled up and the sum of the results recorded were as follows:

10	9	3	6	9	7	4	8	11	12
6	3	11	10	11	9	12	3	9	10
7	11	4	2	6	9	10	5	3	4
4	2	9	9	9	8	12	11	2	5

Prepare a cumulative frequency distribution table for these data without grouping.

3. Use Table 6.8 to determine:
   a) The total number of observations
   b) The width of each class
   c) The class interval of the next class if an additional class was to be added after the last class interval.

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Table 6.8. The Grouped data

Class intervals	40 – 49	50 – 59	60 – 69	70 – 79	80 – 89	90 – 99
Frequencies	7	5	3	2	7	1

4. The community of Mikumi planned to improve the local park. The responses of 18 families who were asked how many times per year they visited the park are shown. Construct a cumulative frequency distribution table with a class width of 2, and the lowest class limit of 20.

20	20	20	21	21	23	24	24	25
25	28	25	26	26	25	28	27	25

5. Given the following frequency distribution table

Table 6.9 Frequency distribution table for grouped data

Class intervals	33 – 36	37 – 40	41 – 44	45 – 48	49 – 52
Frequencies	5	y	11	12	15

a. i. Find the value of y if the total frequency is 50.
   ii. Find the class size of the distribution.

b. Construct the cumulative frequency distribution table.

Presentation of data by graphical and pictorial methods

A graphical representation is a visual or geometrical display of data and statistical results. It is often more effective for interpretation than presenting data in tabular form. There are many different types of graphical representation which are used depending on the nature of the data and the type of statistical information intended or to be displayed.

a) ***Frequency polygon***

The frequency polygon is a line graph that displays the data by connecting the plotted points for the frequencies at the midpoints of the classes. The frequencies are represented by the heights of the points.

Example 6.4: The score of Advanced Mathematics test of 60 students in one of the class at Kishogo Secondary School were recorded as seen in Table 6.10

Table 6.10 Frequency distribution table of marks scored.

Marks	Frequencies
21 – 30	3
31 – 40	4
41 – 50	6
51 – 60	12
61 – 70	15
71 – 80	9
81 – 90	7
91 – 100	4

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Present the data using the frequency polygon by showing all the steps involved.

Solution: Steps for the construction of the frequency polygon

Step 1: Find the class marks.

Step 2: Indicate the points of frequency against the class marks.

Step 3: In plotting the graph, add one class lower than the lowest class and one class higher than the highest class, assign each of them with zero frequency. This is done so that the graph settles on the horizontal axis.

Step 4: Join the points using a ruler starting with the class with zero frequency and end with zero frequency.

Table 6.11 Frequency distribution table of marks scored.

Marks	Class Marks	Frequencies
11 – 20	15.5	0
21 – 30	25.5	3
31 – 40	35.5	4
41 – 50	45.5	6
51 – 60	51.5	12
61 – 70	61.5	15
71 – 80	75.5	9
81 – 90	85.5	7
91 – 100	95.5	4
101 – 110	105.5	0
		N= 60

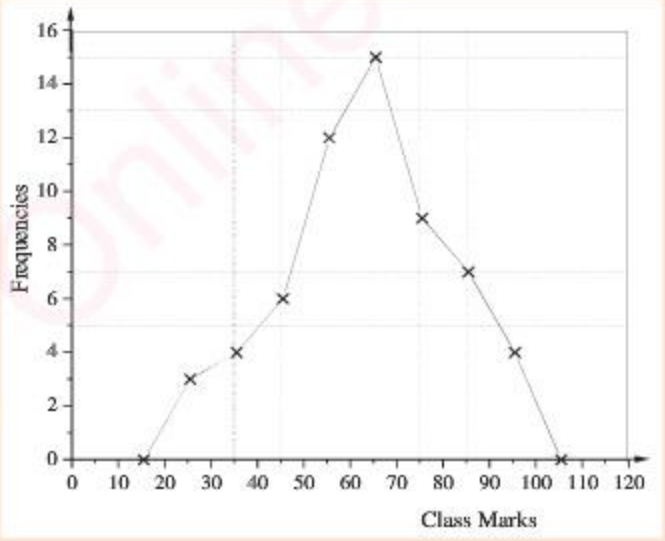

Figure 6.1 The frequency polygon showing the scores for the test.

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b) *Cumulative frequency curve (Ogive)*

A cumulative frequency curve (ogive) is a smooth line graph of cumulative frequency plotted against the upper class boundary.

Example 6.5

Use Table 6.12 to draw a cumulative frequency curve (Ogive)

Table 6.12 Frequency distribution table for grouped data

Length (mm)	11–15	16 – 20	21 – 25	26 – 30	31 – 35	36 – 40	41 – 45
Frequencies (f)	2	4	8	14	6	4	2

Solution

Add a class interval with 0 frequency before the first class and also find the upper boundary for each class interval as shown in Table 6.13.

Table 6.13 A cumulative frequency table.

Length (mm)	**Frequencies** (f)	**Upper boundaries**	**Cumulative frequencies**
6 – 10	0	10.5	0
11 – 15	2	15.5	2
16 – 20	4	20.5	6
21 – 25	8	25.5	14
26 – 30	14	30.5	28
31 – 35	6	35.5	34
36 – 40	4	40.5	38
41 – 45	2	45.5	40
	N = 40		

Plot the cumulative frequency against the upper class boundary of each interval and join the points with a smooth curve as shown in the Figure 6.5.

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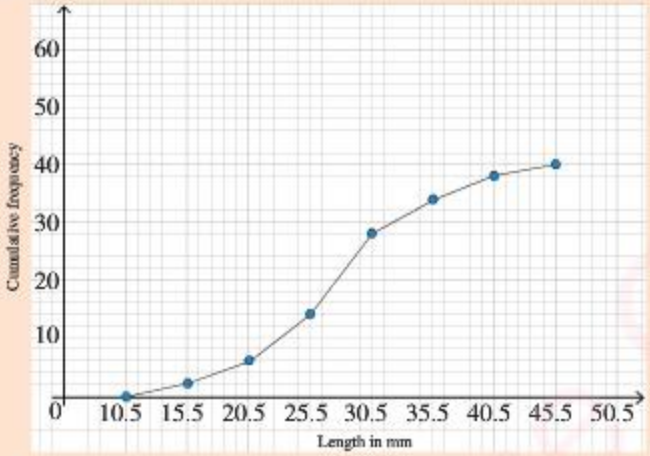

Figure 6.2 A cumulative frequency curve

c) *Histogram*

The histogram is the graph of frequency (vertical axis) against the class marks (horizontal axis) or data values presented by rectangles (blocks). The histogram resembles a bar graph except that when a histogram is drawn, there are no gaps between the blocks as histograms represent continuous data such as temperature while in bar graphs there must be gaps between the blocks as bar graphs are used to represent categorical data such as number of males and females. In plotting histogram the class marks are usually placed at the centre of a rectangle.

Example 6.6

Given the following frequency distribution table:

Table 6.14 Frequency distribution table for grouped data

Marks	Frequencies	Class Marks
21 – 30	3	25.5
31 – 40	4	35.5
41 – 50	6	45.5
51 – 60	12	55.5
61 – 70	15	65.5
71 – 80	9	75.5
81 – 90	7	85.5
91 – 100	4	95.5

a) Draw a histogram b) Draw a bar graph

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Solution

a) Histogram

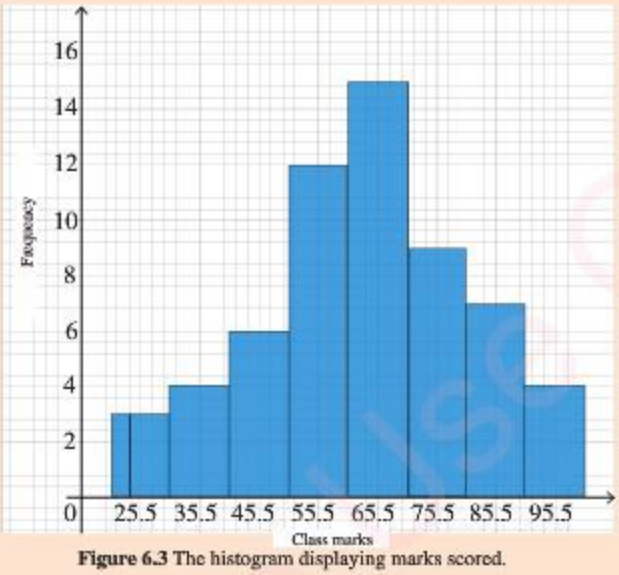

Figure 6.3 The histogram displaying marks scored.

b) A bar graph

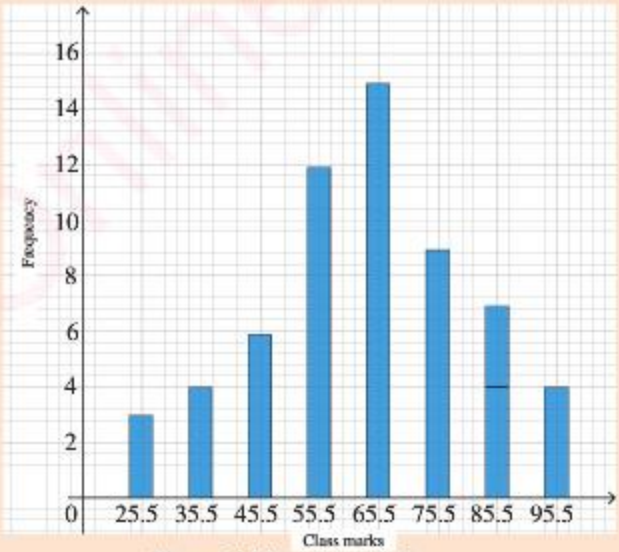

Figure 6.4 A bar graph of marks

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d) **Pictogram**

Pictogram represents the frequency of data as pictures or symbols. Each picture or symbol may represent one or more units of the data as specified based on the nature and quantity of the data.

Example 6.7. Table 6.15 shows the number of cars passing a certain traffic light as were recorded after every 2 hours as follows:

Table 6.15. Number of cars passing traffic lights

Time	06:00 – 08:00	08:00 – 10:00	10:00 – 12:00	12:00 – 14:00	14:00 – 16:00	16:00 – 18:00
Number of cars	150	200	150	100	150	200

Represent the given data using pictogram.

Solution. Let represents 50 cars

Thus:

Figure 6.5 Pictorial representation of cars.

Example 6:8. The number of maize bags produced by a farmer over the past 5 years were recorded as follows:

Table 6.16 Number of maize bags produced by a farmer in 5 years

Years	2012	2013	2014	2015	2016
Number of Bags	200	250	400	200	100

Represent the data using pictogram.

Solution: Let represent 50 bags. Thus:

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Figure 6.6 Pictorial representation of maize bags

Exercise 6.3

1. Table 6.3 shows the number of people visited a certain hospital in one week for treatment.

Table 6.17 Number of patients visited a certain hospital

Ages	0 – 9	10 – 19	20 – 29	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79	80 – 89
Frequencies	84	28	9	24	19	43	63	58	38

   a) Draw a histogram for this frequency distribution table.

   b) Comment on the shape of the graph and give possible reasons for it.

2. The mass of 28 goats to the nearest kg are as follows:

   21 46 21 32 44 47 23
   36 41 36 33 32 22 43
   46 32 26 33 36 28 48
   23 36 36 26 41 28 43

   a) Construct the cumulative frequency distribution table with classes 21–25, 26–30, …

   b) Draw a cumulative frequency curve (Ogive).

3. In a certain region, 17 feeder roads were assessed for further maintenance. Their lengths in kilometres (km) were measured and recorded as follows:

   128.9 32.7 70 50.3 56 145.6 92 97.8 61
   54.3 39.7 79 62.3 55 44 135 87

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a) Construct a frequency distribution table with classes 30 – 49, 50 – 69, …

b) Use the frequency table in (a) above to draw a frequency polygon

4. The score of 30 students in General knowledge test were recorded as follows:

25 26 31 42 19 81 36 51 45 44 54 34 61 52 45

70 86 45 27 78 39 47 90 88 78 41 42 50 43 60

a) Construct the frequency distribution table with classes 11 – 20, 21 – 30, etc.

b) Use the frequency distribution table in (a) to draw the frequency polygon for the scores.

5. A certain school planted 40 trees in 2015 and 90 trees in 2016. In 2017 the school planted 50 trees, while in 2018 the school planted 60 trees. In 2019 the school planted a total of 20 trees. Represent these data by using pictogram.

Presenting data graphically and pictorially using computer spreadsheet package

The statistical data can be presented graphically and pictorially by using an appropriate computer package. In this section, the spreadsheet package (MS Excel) of 2007 will be used to present data graphically as a demonstration. Other packages of spreadsheet can also be applied.

a) Graphical presentation of data using Ms Excel

The following are the procedures:

1. Open an Excel sheet in the computer.
2. Enter the raw data into the spreadsheet. For illustration, the data in Table 6.18 are used to plot the graph.

Table 6.18 Frequency Distribution Table

Class Marks	Frequencies
10	7
15	8
20	4
25	9
30	5

Use column A for class mark ($x-$axis) and column B for the ($y-$axis) frequency.

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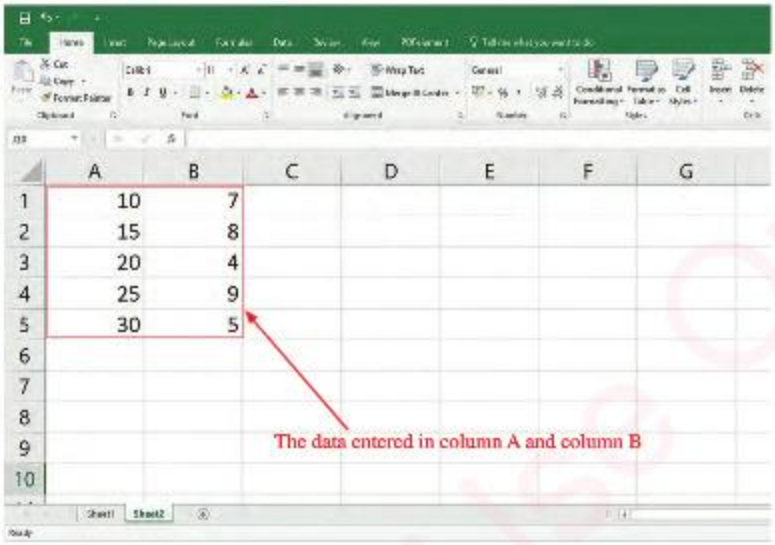

Figure 6.7 Appearance of frequency distribution table into Excel

3. Highlight the data that you wish to graph. Only the highlighted area will be graphed (and nothing else). So, ensure that all the intended data are selected (highlighted).

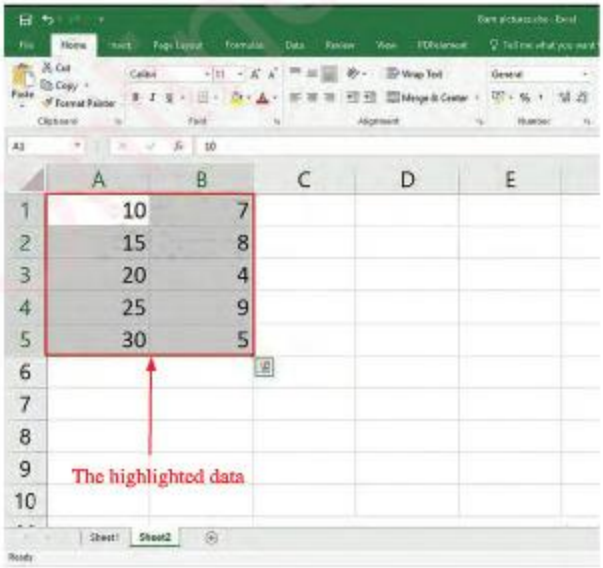

Figure 6.8 Highlighted frequency distribution table into Excel.

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4. With the highlighted data, select the "Insert tab". Then, under the "charts section" of the insert tab, select the type of graph that you want to draw. For this case, choose Bar graph.

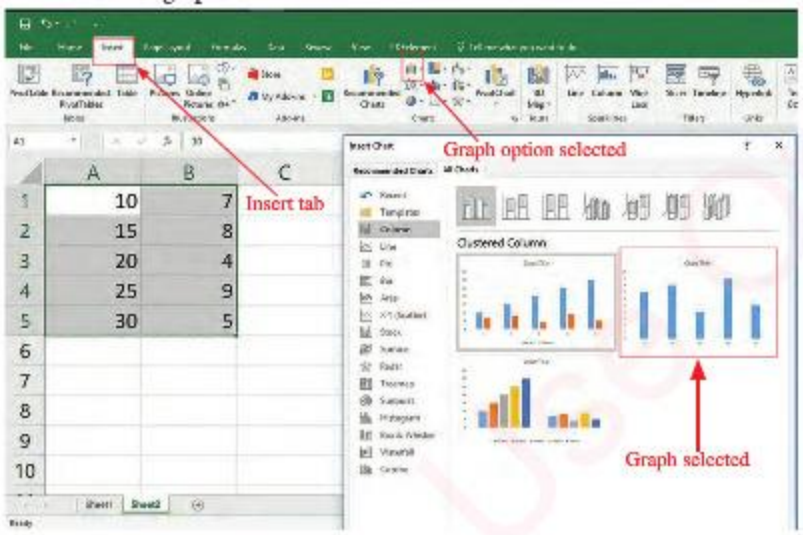

Figure 6.9 Illustration tabs for plotting Bar graph into Excel.

5. Click the type of graph selected, and it will appear in the spreadsheet.

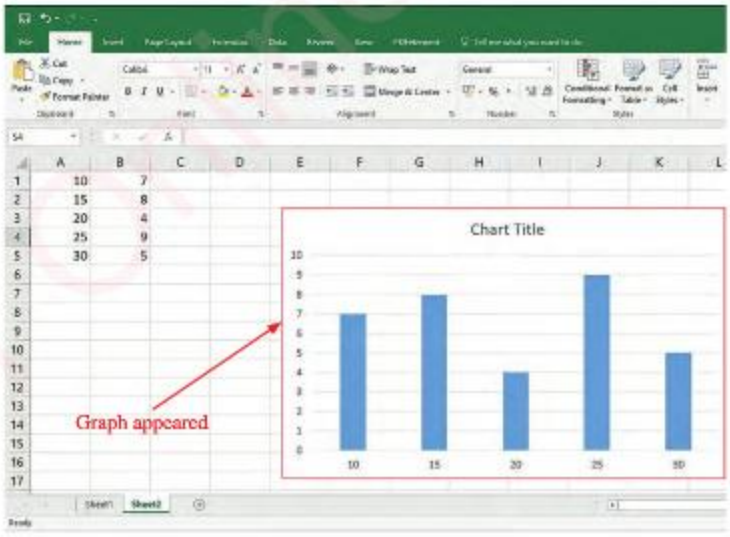

Figure 6.10 Illustration on insertion of the Bar graph into Excel

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6. In order to label the graph under "chart tools", select the "Layout tab". Use the "Axis titles' and "charts titles" buttons to label the graph appropriately. When done with the labelling, the graph should look like the one in Figure 6.11:

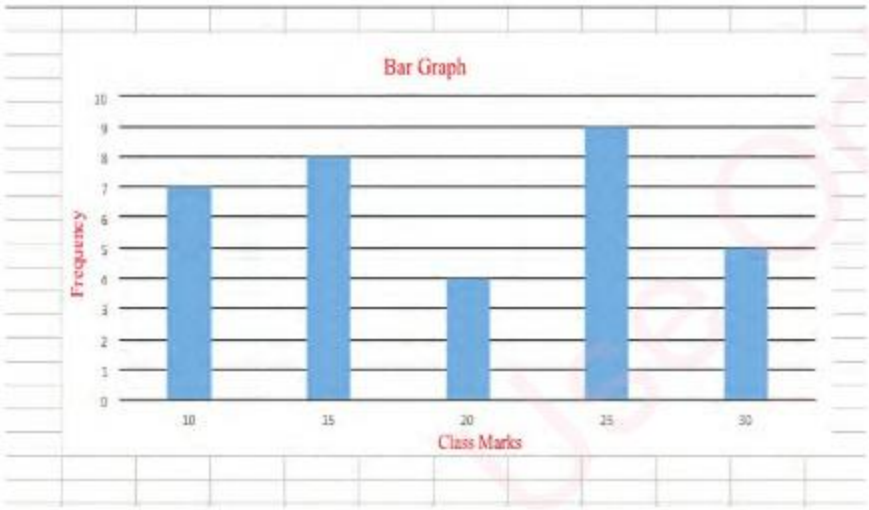

Figure 6.11 Appearance of Bar graph in Excel after labelling.

Remark: The same procedure above can be used to draw any type of graph such as a Histogram and the frequency polygon.

Activity 6.2

Use Computer spreadsheet package to draw on separate sheets a bar graph and a histogram for the following distribution.

Table 6.19 Scores of a certain Mathematics test.

Scores (x)	10	20	30	40	50	60	70	80	90	100
Frequencies	8	10	5	4	12	15	9	20	7	10

b) Pictorial presentation of data using Ms Excel

Pictorial presentation into Excel can be done using the following procedures:

1. Open an Excel sheet in the computer
2. Enter the raw data into the spreadsheet. For illustration, use the data in Table 6.20:

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Table 6.20 Sample data for drawing in an Excel sheet

Stationery	Number of stationeries
Pencils	6
Books	8
Laptops	5
Printers	2
Staplers	5

Use column A for x-axis i.e. stationeries and column B for the dependent variable (y-axis) i.e. number of stationeries.

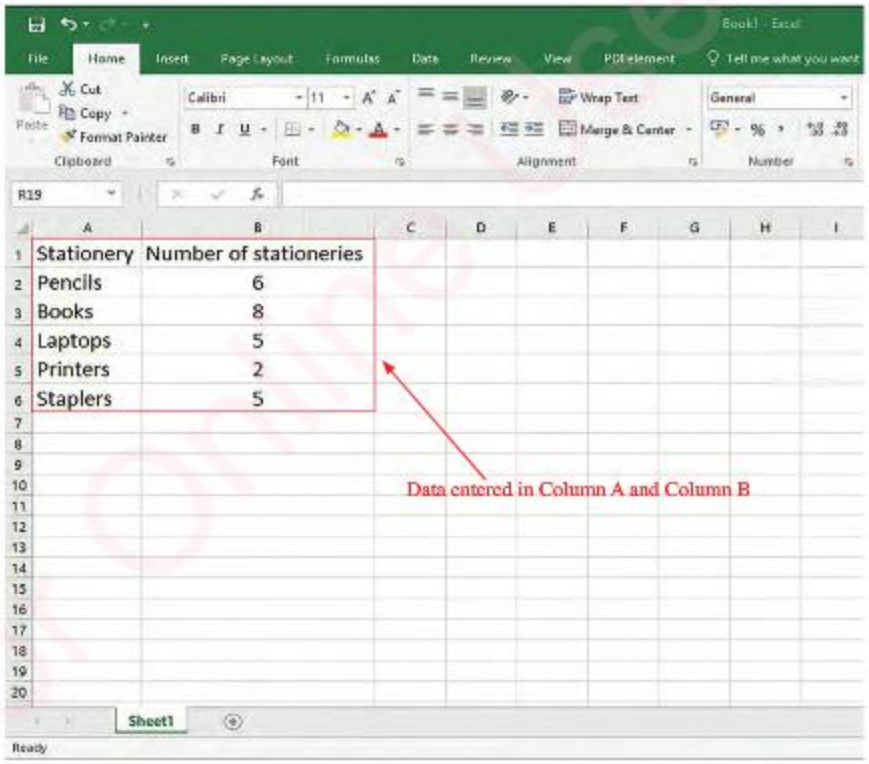

Figure 6.12 Data entry stage into Excel.

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3. Highlight the data that you wish to graph. Only the area highlighted will be graphed (and nothing else). Hence ensure that all the intended data are selected.

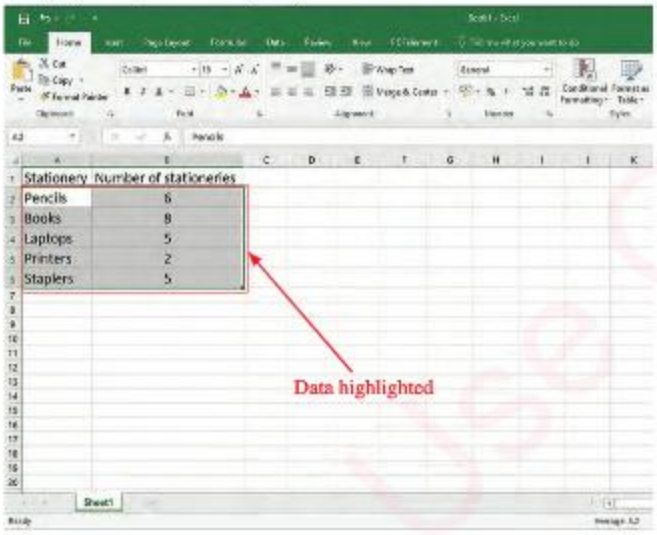

Figure 6.13 Highlighted data into Excel.

4. With the highlighted data, select the "Insert tab". Then, under the "chart section" of the "insert tab", select the type of graph to be drawn. For this case choose Bar graph.

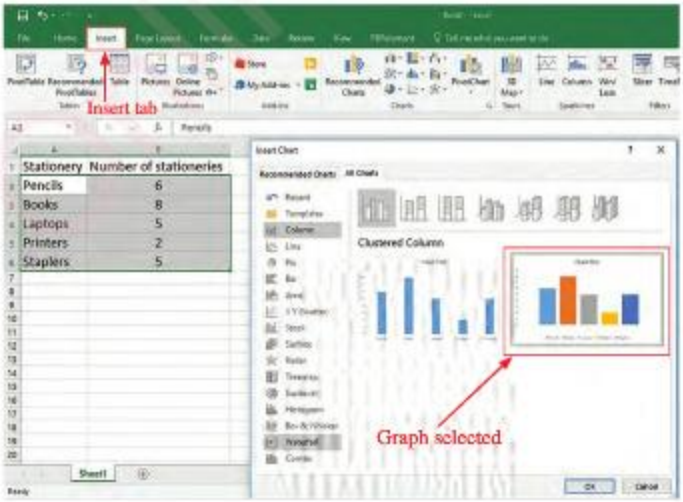

Figure 6.14 Illustration on plotting pictograph into Excel.

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5. Click the type of graph selected, and it will appear in the spreadsheet.

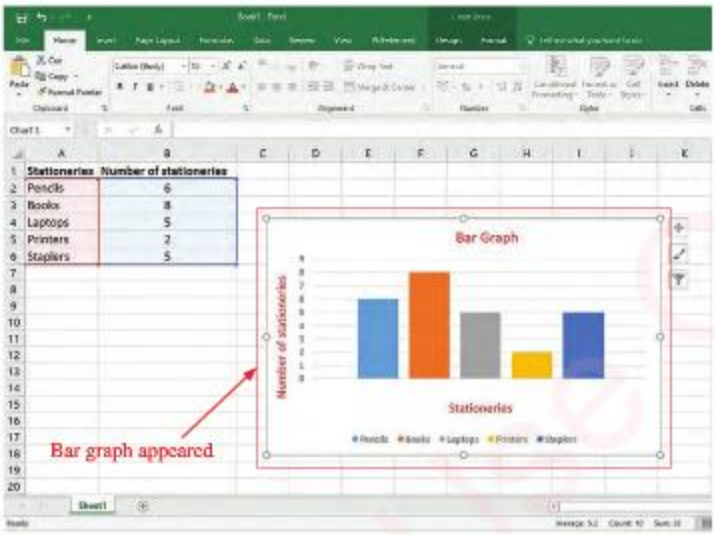

Figure 6.15 Appearance of Bar Graph in Excel

6. Creation of a pictograph is achieved by adding a picture instead of bars in the drawn graph. For this, select the first bar in the chart, click the bar two times to select it exclusively as shown in Figure 6.16:

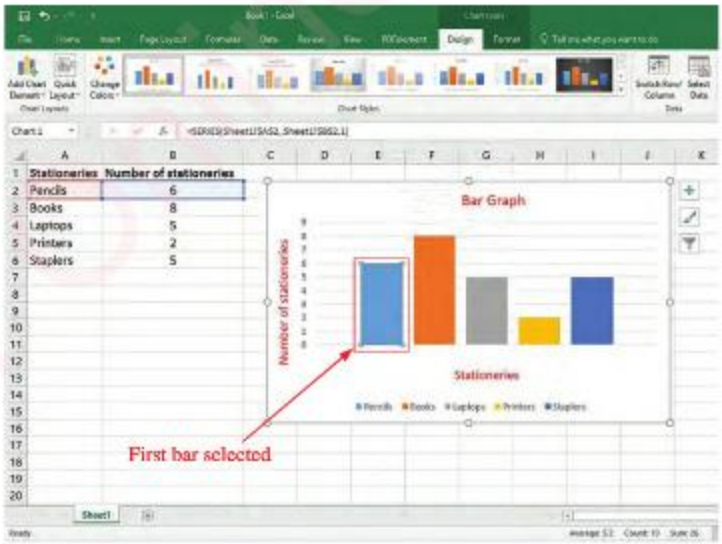

Figure 6.16 Illustration on how to select a single bar graph in Excel

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7. Right click the first selected bar graph, and then follow the steps in Figure 6.17 to insert the picture instead of bar graph.

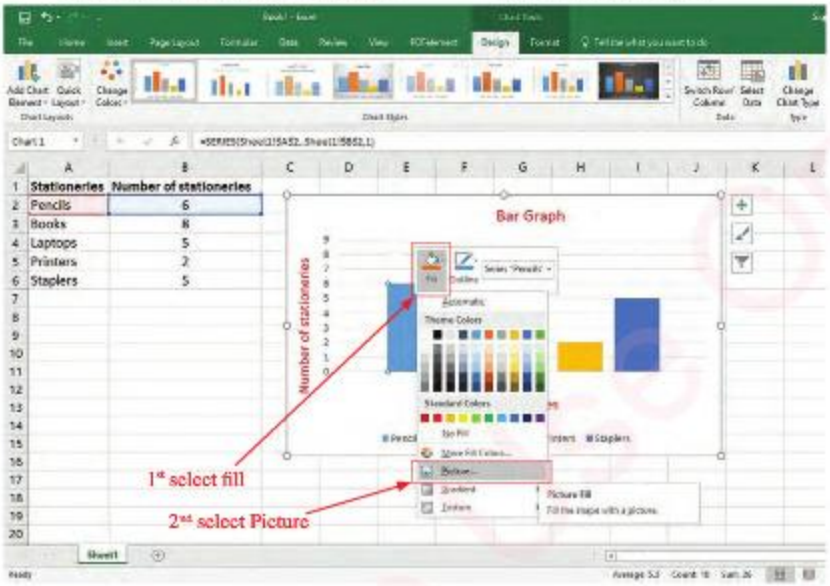

Figure 6.17 Inserting picture to pictograph in Excel.

8. From the 'Picture' pane, search for pencil picture. Click the picture of the pencil to be included and then click "insert".

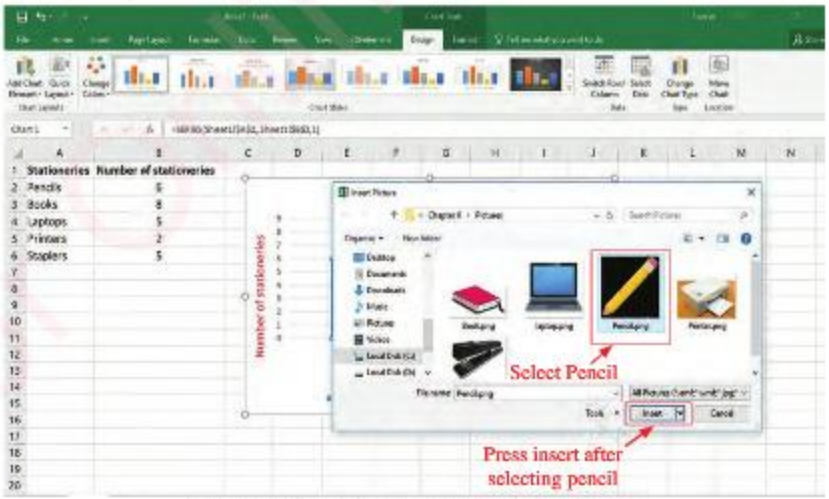

Figure. 6.18 Illustration on selection of pencil drawing

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9. After pressing the insert, the pencil picture will appear instead of the bar graph as shown in the Figure 6.19.

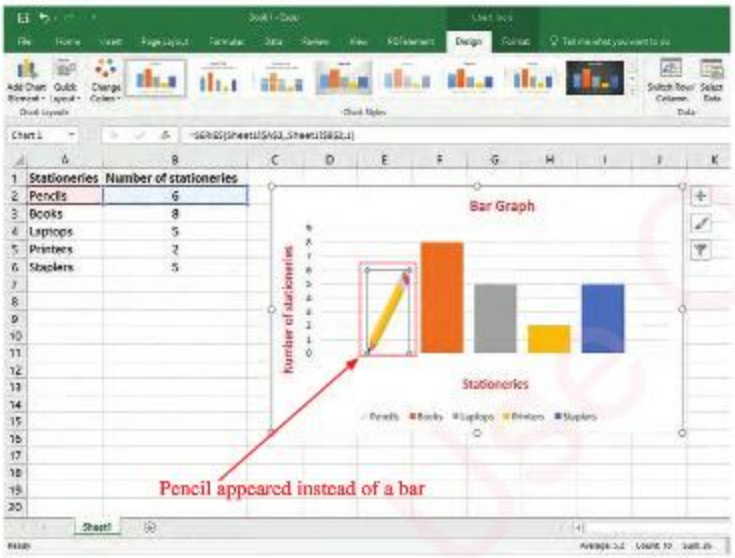

Figure 6.19 Appearance of object instead of a Bar.

In order to show the number of pencils in the graph by depicting the same number of pencil images, right click the stretched image and click format data series.

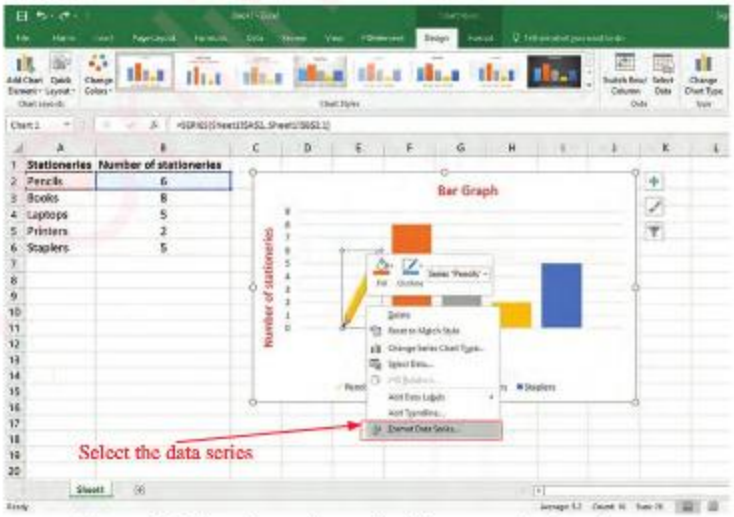

Figure 6.20 Illustration on formatting objects to replace bar charts.

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10. After clicking format data series, "series options" will appear. Select "Fill", then enable "Stack and scale" with option. Follow the procedure as shown in Figure 6.15.

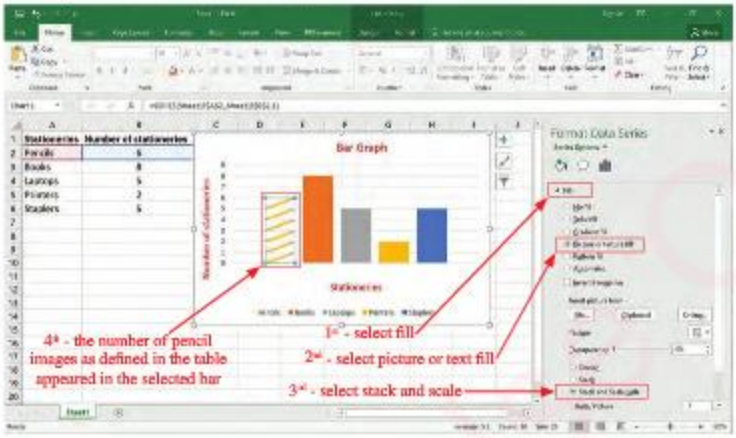

Figure 6.21 Illustration on insertion of objects.

11. Click "Close" after the above10th step of the procedure. Then, the number of pencil images as defined in the table will be shown in Figure 6.21. Now, repeat the steps 6 up to 11 of the procedure for the other bars to place suitable images. Lastly, the pictorial bar graph will appear as shown in Figure 6.22.

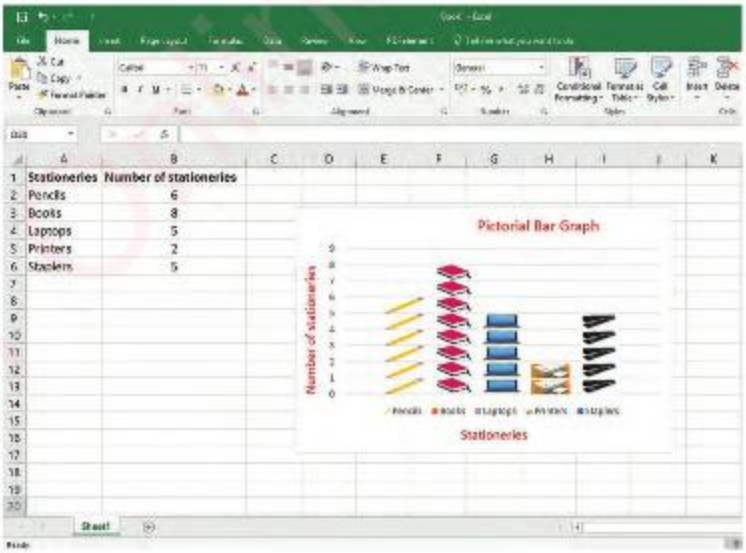

Figure 6.22 Pictorial representation of objects into Excel.

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Activity 6.3

Use Computer spreadsheet package to draw a pictorial representation of the following group of data:

Table 6.21 Office equipments

Stationery	Laptop	Printer	Book	Pencil	Stapler	Glue
Number of stationeries	10	5	6	4	7	3

6.2 Measures of central tendency for grouped and ungrouped data

Measures of central tendency are quantitative measures that are intended to capture the value of data at the centre as a representative of the given data set. These are statistical measure that identifies a single score as representative of the entire data set. A measure of central tendency is a value somewhere between the minimum and the maximum value and it can better represent the data or the distribution for unimodal and symmetric distribution. In this section, five measures of central tendency will be discussed which are mean, median, mode, quartile and percentile.

Remark: Quartiles, deciles and percentiles are mostly known as measures of position, but also considered as for central tendency because to some cases are equal to the value of median.

The Mean

The mean (Arithmetic mean) refers to the average of scores in the data, denoted by $\bar{x}$ for the scores $x_1, x_2, x_3, \ldots, x_n$

Therefore,

$$\bar{x} = \frac{\text{sum of values}}{\text{Total number of values}} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{N}$$

a) Mean for ungrouped data:

Using sigma notation $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{N}$. (6.1)

When the values $x_1, x_2, x_3, \ldots, x_n$ occur with frequencies $f_1, f_2, f_3, \ldots, f_n$, then the mean is calculated as follows:

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + f_3x_3 + \ldots + f_nx_n}{f_1 + f_2 + f_3 + \ldots + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i} \quad (6.2)$$

Thus the equations (6.1) and (6.2) are appropriate formulae to compute the mean for ungrouped data.

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Example 6.9

The height of each of the five randomly selected students in a certain primary school was recorded in centimetres as follows: 80, 90, 100, 110, 120. Find the mean height of the students.

Solution

Use $\bar{x} = \frac{\sum_{i=1}^{5} xi}{N}$

$$= \frac{80+90+100+110+120}{5} = \frac{500}{5} = 100$$

$\therefore$ Mean$(\bar{x}) = 100$ cm

Example 6.10

Find the mean of the following ungrouped data

Table 6.22 Table of ungrouped data

Scores	Frequencies
4	2
8	1
11	2
13	5
15	4
17	1

Solution

Use $\bar{x} = \frac{\sum_{i=1}^{6} f_i x_i}{\sum_{i=1}^{6} f_i}$

$$\bar{x} = \frac{(4\times2)+(8\times1)+(11\times2)+(13\times5)+(15\times4)+(17\times1)}{2+1+2+5+4+1}$$

$$= \frac{180}{15}$$

$\therefore \bar{x} = 12$

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b) *The Mean for Grouped Data*

i) **Common mean method:**

The mean of grouped data can be computed differently from the ungrouped data. For given grouped data, class marks are used in computation of the mean.

Thus,
$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} \tag{6.3}$$

where x_i represents class marks, n represents total number of data entries, and f_i represents frequency of the class.

Example 6.11 The following frequency distribution table shows the hypothetical data of the marks for Basic Applied Mathematics test of a Form five class at Kiponzelo Secondary School. Use this distribution table to compute the mean score.

Table 6.23 Marks of BAM test.

Marks	Frequencies
15 – 19	6
20 – 24	11
25 – 29	9
30 – 34	8
35 – 39	2
40 – 44	3
45 – 49	1

Solution

Table 6.24 Solution for problem in Table 6.23.

Class intervals	Class Marks (x_i)	Frequencies (f_i)	($f_i x_i$)
15 – 19	17	6	102
20 – 24	22	11	242
25 – 29	27	9	243
30 – 34	32	8	256
35 – 39	37	2	74
40 – 44	42	3	126
45 – 49	47	1	47
	$\sum_{i=1}^{7} f_i = 40$		$\sum_{i=1}^{7} f_i x_i = 1090$

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Therefore, Mean

$$\bar{x} = \frac{\sum_{i=1}^{7} f_i x_i}{\sum_{i=1}^{7} f_i} = \frac{1090}{40} = 27.25$$

Assumed mean method

The method uses an assumed mean (guessed mean) in order to calculate the actual mean.

Let A be an assumed mean and d_i be the deviation of each data point or class mark from A, then $d_i = x_i - A$

$$\therefore x_i = A + d_i$$

Substituting $x_i = A + d_i$ in the formula

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

Thus, $$\bar{x} = \frac{\sum_{i=1}^{n} f_i (A + d_i)}{\sum_{i=1}^{n} f_i}$$

$$= \frac{\sum_{i=1}^{n} f_i A}{\sum_{i=1}^{n} f_i} + \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$$

$$= \frac{A\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} f_i} + \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i},$$

since A is constant.

Now, $$\bar{x} = A + \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$$

$$\therefore \bar{x} = A + \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}.$$

$N = \sum_{i=1}^{n} f_i$ represents the sum of the frequencies in the distribution, and

$\sum_{i=1}^{n} f_i d_i$ represents sum of product of the frequencies and corresponding deviation from the respective class mark in given class interval.

Example 6.12 Compute the mean of the following data recorded after the form two terminal examinations at a certain Secondary School using assumed mean method.

Table 6.25 Grouped data

Marks	Number of students
10 – 20	4
20 – 30	5
30 – 40	4
40 – 50	5
50 – 60	3
60 – 70	4
70 – 80	16
80 – 90	13
90 – 100	11

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Solution: Let Assumed mean, $A = 55$

Table 6.26 Solution for problem in Table 6.26.

Class Intervals	Class marks $s(x_i)$	Frequencies (f_i)	$d_i = x_i - A$	$f_i d_i$
10 – 20	15	4	15 – 55 = –40	–160
20 – 30	25	5	25 – 55 = –30	–150
30 – 40	35	4	35 – 55 = –20	–80
40 – 50	45	5	45 – 55 = –10	–50
50 – 60	55	3	55 – 55 = 0	0
60 – 70	65	4	65 – 55 = 10	40
70 – 80	75	16	75 – 55 = 20	320
80 – 90	85	13	85 – 55 = 30	390
90 – 100	95	11	95 – 55 = 40	440
		$\sum_{i=1}^{9} f_i = 65$		$\sum_{i=1}^{9} f_i d_i = 750$

$$\bar{x} = A + \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$$

$$= 55 + \frac{750}{65}$$

$$= 55 + 11.54$$

$$\therefore \quad \bar{x} = 66.54$$

ii) **Mean by coding method**

In this method, the mean is evaluated by organising the original data for simple and easier calculations. The coding method finds the mean for grouped data from the assumed mean using unit deviations. By coding method, the mean is calculated using the following formula:

$$\bar{x} = A + \frac{c\left(\sum_{i=1}^{n} f_i u_i\right)}{\sum_{i=1}^{n} f_i}$$

where;

A represents assumed mean.

c represents class size (width).

$u_i = \frac{d_i}{c}$ represents deviation per class size

$d_i = x_i - A$.

$f_i =$ represents frequency.

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Example 6.13 Consider the following distribution table which shows the weights (kg) of pregnant women attending clinic at a certain hospital.

Table 6.27 Women attending clinic.

Class intervals	Frequency
60 – 62	5
63 – 65	18
66 – 68	42
69 – 71	27
72 – 74	8

Find the mean of the given data using coding method and the class marks of the class interval 66 – 68 as assumed mean.

Solution From $A = \frac{66+68}{2} = 67$ and $c = 3$

Table 6.28 Women attending clinic (solution for problem in Table 6.27).

Class Intervals	Class Marks (x_i)	Frequencies f_i	$d_i = x_i - A$	$u_i = \frac{d_i}{c}$	$f_i u_i$
60 – 62	61	5	–6	–2	–10
63 – 65	64	18	–3	–1	–18
66 – 68	67	42	0	0	0
69 – 71	70	27	3	1	27
72 – 74	73	8	6	2	16
		$\sum_{i=1}^{5} f_i = 100$			$\sum_{i=1}^{5} f_i u_i = 15$

$$\bar{x} = A + \frac{c\left(\sum_{i=1}^{n} f_i u_i\right)}{\sum_{i=1}^{n} f_i}$$

$$= 67 + 3\left(\frac{15}{100}\right)$$

$$= 67 + \frac{45}{100}$$

$$= 67 + 0.45 = 67.45$$

$$\therefore \bar{x} = 67.45$$

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Exercise 6.4

1. Find the mean of each of the following set of data of the planted types of trees in a particular village.
   a) 6, 7, 10, 12, 13, 4, 8, 12
   b) 22, 24, 30, 27, 29, 31, 25, 28, 41, 42
2. Calculate the mean from each of the following set of data:
   a) 1, 2, 3, 4, 5, 6, 7, 8, 9
   b) 10, 20, 30, 40, 50, 60, 70, 80, 90, 100
3. The scores of Additional Mathematics test in a certain school is given in Table 6.29.

Table 6.29 Scores of Additional Mathematics test

Scores	40 – 44	45 – 49	50 – 54	55 – 59	60 – 64	65 – 69
Frequencies	8	11	14	9	2	1

Calculate the mean:

a) By using assumed mean method
b) By using coding method, taking $A = 52$
c) Comment on the solution of (a) and (b)

4. Compute the mean of the following distribution

Table 6.30 Grouped data

Class intervals	0 – 4	5 – 9	10 – 14	15 – 20	21 – 24	25 – 30
Frequencies	1	14	23	21	15	6

5. The following data relates to the number of successful sales made by the salesmen employed by a large microcomputer firm in a particular quarter.

Table 6.31 Sales made by salesmen in a microcomputer firm

Number of sales	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50	51 – 60
Number of Salesmen	3	16	26	31	16	8

Calculate the mean of the sales

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The Median

The median is a middle value of the set of given data if data are arranged in ascending or descending order. It divides the remaining arrayed data into two equal parts.

a) Median for ungrouped data

If the total number of data is odd, the median is the $\frac{(N+1)^{th}}{2}$ value.

If the total number of data is even, the median is the $\frac{\left(\frac{N}{2}\right)^{th}+\left(\frac{N}{2}+1\right)^{th}}{2}$ value where N is the number of given values.

Example 6.14

Find the median of the values 27, 25, 19, 23, 31, 42, 51, 17, 22

Solution

Here $N=9$ and since N is odd, the median is the $\left(\frac{9+1}{2}\right)^{th}=5^{th}$ value.

Arrange data in ascending order:

17, 19, 22, 23, 25, 27, 31, 42, 51

$\therefore$ Median=25

Example 6.15

Find the median of the values 2, 5, 5, 7, 12, 13, 15, 15, 16, 19

Solution

From the data, $N=10$ (even number)

$$=\frac{\left(\frac{10}{2}\right)^{th}+\left(\frac{10}{2}+1\right)^{th}}{2}=\frac{5^{th}+6^{th}}{2}$$

Hence, the median here is between 5^{th} and 6^{th} values. So, median is the average of the two values.

Now, 5^{th} value $=12$

6^{th} value $=13$

Hence, the median $=\frac{12+13}{2}=12.5$

$\therefore$ Median $=12.5$

Example 6.16

Find the median score if the score x occurs with frequencies as given in Table 6.32.

Table 6.32 Frequency distribution table.

Scores (x)	Frequencies
1	1
2	3
3	2
4	4
5	8
6	2

Solution

Table 6.33 Solution to Table 6.32.

Scores	Frequencies	Cumulative frequencies
1	1	1
2	3	4
3	2	6
4	4	10
5	8	18
6	2	20

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$$\text{Median} = \frac{\left(\frac{20}{2}\right)^{th} + \left(\frac{20}{2}+1\right)^{th}}{2} = \frac{10^{th} + 11^{th}}{2}$$

The median is between the 10^{th} and 11^{th} value and it is an average of the two numbers $\text{Median} = \frac{4+5}{2} = 4.5$

$\therefore$ Median = 4.5

b) ***Median for grouped data***

The computation of median in grouped data will involve several items in class intervals. The median value of a grouped data lies at the class where the $\left(\frac{N}{2}\right)^{th}$ position belongs.

Median class is computed by using the formula;

$$\text{Median} = L + \left(\frac{\frac{N}{2} - f_b}{f_w}\right)c$$

where L represent the lower class boundary of the median class at the $\left(\frac{N}{2}\right)^{th}$ position.

f_b represents sum of the frequencies for the classes lower than the median class (ascending order class intervals).

f_w represents frequency of the median class.

$N = \sum_{i=1}^{n} f_i$ represents total number of observation.

c represents class size (width).

Example 6.17: The following frequency distribution tables show the data of age of children in months in a certain street.

Table 6.34 Age of children

Ages (months)	Frequencies
0 – 8	10
9 – 17	5
18 – 26	15
27 – 35	3
36 – 44	1

Table 6.35 Age of children

Ages (months)	Frequencies
10 – 15	30
15 – 20	42
20 – 25	20
25 – 30	23
30 – 35	13
35 – 40	8
40 – 45	4

Find the median age of the children in;

a) Table 6.34 b) Table 6.35

Solution

a)

Table 6.36 Age of children

Ages	Frequencies	Cumulative frequencies
0 – 8	10	10
9 – 17	5	15
18 – 26	15	30
27 – 35	3	33
36 – 44	1	34

Median position $= \frac{N}{2} = \frac{34}{2} = 17^{th}$ position.

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The median class is 18 – 26 as per cumulative value of 17th position.

Now, $L=17.5, N=34,$

$f_b=15, f_w=15, c=9$

$$\text{Median}=L+\left(\frac{\frac{N}{2}-f_b}{f_w}\right)c$$

$$=17.5+\left(\frac{\frac{34}{2}-15}{15}\right)\times 9$$

$$=17.5+1.2=18.7$$

$$=18.7$$

Therefore, median age of children is 18.7

b)

Table 6.37 Solution to the problem in Table 6.36

Ages	Frequencies	Cumulative frequencies
10 – 15	30	30
15 – 20	42	72
20 – 25	20	92
25 – 30	23	115
30 – 35	13	128
35 – 40	8	136
40 – 45	4	140

Median position $=\frac{N}{2}=\frac{140}{2}=70^{th}$ position.

The median class is 15 – 20 (from 72 cumulative value)

Now, $L=14.5, f_b=30, f_w=42, c=6$

$$\text{Median}=L+\left(\frac{\frac{N}{2}-f_b}{f_w}\right)c$$

$$=14.5+\left(\frac{70-30}{42}\right)\times 6$$

$$=14.5+5.71$$

$$=20.2$$

Therefore, the median age of children is 20.2.

Exercise 6.5

1. Calculate the median for the set of values: 43, 75, 48, 51, 51, 47, 50
2. Calculate the median for the following distribution of orders in 12 days by milk production firm.

Table 6.38 Delivery of orders by the milk production firm

Delivery days	1	2	3	4	5	6	7	8	9	10	11	12
Number of orders	4	8	11	12	21	15	10	4	2	2	1	1

3. Estimate the median age for the following data which represents the ages of a set of 130 representatives who took part in a statistical survey.

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Table 6.39 Ages of representative of statistical survey

Ages in years	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44	45 – 49
Number of representations	2	14	29	43	33	9

4. Eighty students did an examination where the maximum pass possible was 100 marks, and the following were the results:

Table 6.40 Examination results

Marks	10 – 19	20 – 29	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79	80 – 89
Frequencies	1	8	12	20	18	10	7	4

Find.

a) The class-width of the classes.

b) Calculate the median.

5. The length of pieces of iron rods (in cm) in a certain hardware were recorded as:
42.0, 52.3, 55.2, 72.9, 52.8, 79.0, 32.5, 15.2, 27.9, 30.2
Calculate the median length.

The Mode

Mode is a value in the data set that occurs most frequently. A distribution may have one mode, two modes, three modes etc. The mode is used when the most typical case is desired. It is also used when the data are nominal such as religions preference, gender or political affiliation. In business, mode can be applied in identifying preffered commodity by customers of certain place.

a) The Mode of ungrouped data

Example 6.18

Find the mode of the values:

a) 9, 8, 10, 10, 10, 12, 6, 11, 12, 10, 8, 8

b) 110, 120, 70, 90, 90, 100, 80, 130, 40

Solution

a) **Table 6.41** Showing the frequency distribution table of data in Example 18(a).

Values	6	8	9	10	11	12
Tallies	/	///	/	////	/	/
Frequencies	1	3	1	4	1	2

Therefore, a value with highest frequency is 10 with a frequency of 4. Hence the mode is 10.

b) **Table 6.42** Frequency distribution table of data in Example 18(b).

Values	40	70	80	90	100	110	120	130
Tallies	/	/	/	/	/	/	/	/
Frequencies	1	1	1	2	1	1	1	1

Therefore, a value with highest frequency is 90. Hence the mode is 90.

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b) The Mode for grouped data

Mode for the grouped data is computed by the formula:

$$\text{Mode} = L + \left(\frac{d_1}{d_1 + d_2}\right) c$$

where;

L represents the lower class boundary of the modal class.

d_1 represents the difference between the modal class frequency and the frequency of the next lower class (ascending order of class intervals).

d_2 represents the difference between the modal class frequency and the frequency of the next higher class.

c represents the class size

Example 6.19 Find the mode from the following frequency distribution table:

Table 6.43 Grouped data

Class intervals	Frequencies
15 – 19	6
20 – 24	11
25 – 29	9
30 – 34	8
35 – 39	2
40 – 44	3
45 – 49	1

Solution

Use the formula:

$$= L + \left(\frac{d_1}{d_1 + d_2}\right) c$$

Modal class is 20 – 24, and note that

$L = 19.5, c = 5, \quad d_1 = 11 - 6 = 5,$

and $d_2 = 11 - 9 = 2$

$$\text{Mode} = 19.5 + \left(\frac{5}{2+5}\right) \times 5$$

$$= 19.5 + \left(\frac{5}{7}\right) \times 5$$

$$\text{Mode} = 19.5 + 3.57$$

$$\text{Mode} = 23.07$$

$\therefore$ The mode is 23.07

Example 6.20 Find the modal class and the mode from the following frequency distribution table.

Table 6.44 Frequency distribution table of grouped data

Class intervals	Frequencies
300 – 320	32
321 – 340	30
341 – 360	42
361 – 380	22
381 – 400	20
401 – 420	35
421 – 440	40
441 – 460	38
461 – 480	41

Solution

The modal class is (341 – 360), with the highest frequency of 42.

$$\text{Mode} = L + \left(\frac{d_1}{d_1 + d_2}\right) c,$$

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note that $L = 340.5$, $d_1 = 42 - 30 = 12$,

$d_2 = 42 - 22$, and $c = 360.5 - 340.5 = 20$

$$\text{Mode} = 340.5 + \left(\frac{12}{12+20}\right)20$$

$$= 340.5 + \frac{224}{32} = 348$$

$\therefore$ Mode $= 348$

Exercise 6.6

1. Find median and mode of each of the following sets of data.
   a) 2, 3, 6, 10, 9, 2, 8, 3, 11, 2, 10, 12
   b) 10, 20, 10, 30, 50, 40, 10, 20, 20, 60, 30

2. The number of students in a certain school with ages between 10 and 16 years were recorded as shown in Table 6.45.

Table 6.45 Ages of students

Ages	10	11	12	13	14	15	16
Number of students	2	8	16	10	14	7	3

Find:

a) mean b) median c) mode

3. The marks of History test in a class of 100 students were recorded as presented in Table 6.46.

Table 6.46 History test scores

Marks	30 – 39	40 – 49	50 – 69	60 – 69	70 – 79	80 – 89
Number of students	7	18	$2x$	x	12	3

Calculate: a) The value of x b) Median c) Mode

4. Table 6.47 shows the marks scored by 80 students in a Basic Applied Mathematics examination.

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Table 6.47 BAM examination results

Marks	Frequencies
10 – 19	1
20 – 29	8
30 – 39	12
40 – 49	20
50 – 59	18
60 – 69	10
70 – 79	7
80 – 89	4

a) Find the class width of the classes
b) Calculate the median

5. Consider the given findings in the Table 6.48.

Table 6.48 Frequency distribution table of grouped data

Class intervals	0 – 4	5 – 9	10 – 14	15 – 20	21 – 24	25 – 29	30 – 34
Frequencies	3	8	11	9	4	5	2

Find a) Modal class b) Mode

c) Explain the applications of mode in real life situation such as in business, health, education, etc.

Estimation of Mode using histogram

The mode of data can be estimated from the drawn histogram as follows:

1. Draw a straight line from top left corner of the tallest rectangle to the top left corner of the rectangle in the right of it.
2. Draw a straight line from top right corner of the tallest rectangle to the top right corner of the rectangle which is in the left hand side.
3. Observe where these lines intersect.
4. Mode is the point of intersection which can be read on the horizontal axis.

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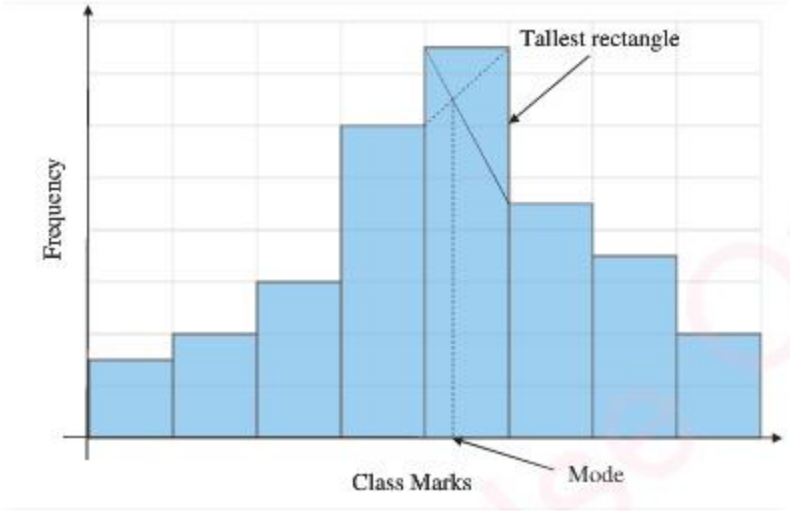

Figure 6.23 Illustration for an estimation of mode using histogram

Quartiles and percentiles of the data

a) *The Quartiles*

Quartiles are obtained when the given data is divided into four equal parts (quarters) when arranged in ascending order. These are first quartile Q_1, second quartile Q_2 and third quartile Q_3. The first (lower) quartile Q_1 is defined as the middle value between the smallest value and the value before the median value of the data given. The Q_1 divides data to be 25% below Q_1 and 75% of data to be above Q_1. The second (middle) quartile is the median value of the data. Q_2 divides data to be 50% below and 50% above Q_2.

The third (upper) quartile Q_3 is the middle value between the value after the median value and the highest value of the given data 25% above Q_3 and 75% of data below Q_3. Quartiles are used as a measure of positions and also central tendency.

i) **Quartile for ungrouped data.**

The position of the first quartile Q_1

$$Q_1 = \left(\frac{N+1}{4}\right)^{th} \text{ value if N is odd and}$$

$$Q_1 = \left(\frac{N}{4}\right)^{th} + \left(\frac{N}{4}+1\right)^{th} \text{ value if N is even.}$$

The position of the second quartile

$$Q_2 = \left(\frac{2(N+1)}{4}\right)^{th} = \left(\frac{N+1}{2}\right)^{th} \text{ value if N is odd and}$$

$$Q_2 = \frac{\left(\frac{2N}{4}\right)^{th} + \left(\frac{2N}{4}+1\right)^{th}}{2} = \frac{\left(\frac{N}{2}\right)^{th} + \left(\frac{N}{2}+1\right)^{th}}{2}$$

value if N is even.

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The position of the third quartile

$Q_3 = \left(\frac{3}{4}(N+1)\right)^{th}$ value if N is odd and

$Q_3 = \frac{\left(\frac{3N}{4}\right)^{th} + \left(\frac{3N}{4}+1\right)^{th}}{2}$ value if N is even.

Example 6.21

Find the first and the third quartiles of the data set 3, 7, 8, 5, 12, 14, 21, 13, 18.

Solution

First, write the data set in ascending order as $\{3,5,7,8,12,13,14,18,21\}$

Now, $\{\underbrace{3,5,7,8}_{\text{Lower half}}, \underbrace{12}_{\text{Median}}, \underbrace{13,14,18,21}_{\text{Upper half}}\}$

Hence, the first quartile is the median of the lower half of the data which is $\{3,5,7,8\}$

Since there is an even number of values, need to find the mean of the middle two values in order to find the first quartile:

That is, $Q_1 = \frac{5+7}{2} = \frac{12}{2} = 6$

Similarly, the third quartile is the median of the upper half of the data which is $\{13,14,18,21\}$

Since there is an even number of values, find the mean of the two middle values in order to find the third quartile:

That is, $Q_3 = \frac{14+18}{2} = \frac{32}{2} = 16$

Therefore, $Q_1 = 6$ and $Q_3 = 16$

ii) **Inter-quartile range**

The difference between the first and third quartile is called inter-quartile range. It is equal to the third quartile minus the first quartile. Inter-quartile range is also a measure of variability.

That is, $I.Q.R = Q_3 - Q_1$

iii) **Semi inter-quartile range**

This is the half of inter-quartile range,

That is, semi inter-quartile range

$= \frac{1}{2}$ inter-quartile range $= \frac{1}{2}(\ _3 - \ _1)$

Example 6.22

Find the median, lower quartile, upper quartile and interquartile range of the following numbers: 14, 23, 39, 4, 11, 19, 45, 31, 7, 30, 42, 17.

Solution:

Arranging the data in ascending order, leads to:

Table 6.49 Location of quartiles

4	7	11	▲	14	17	19	▲	23	30	31	▲	39	42	45
			Lower quartile				Median				Upper quartile			

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lower quartile $= \frac{11+14}{2} = 12.5$

The median $= \frac{19+23}{2} = 21$

upper quartile $= \frac{31+39}{2} = 35$

Inter-quartile range is upper quartile – lower quartile $= 35 - 12.5 = 22.5$

Alternative method;

Since the total number of data is even, then position of

$$Q_1 = \frac{\left(\frac{N}{4}\right)^{th} + \left(\frac{N}{4}+1\right)^{th}}{2}$$

$$= \frac{\left(\frac{12}{4}\right)^{th} + \left(\frac{12}{4}+1\right)^{th}}{2} = \frac{3^{rd}+4^{th}}{2} \text{ value.}$$

$$Q_1 = \frac{11+14}{2} = 12.5$$

The position of $Q_3 = \frac{\left(\frac{3N}{4}\right)^{th} + \left(\frac{3N}{4}+1\right)^{th}}{2}$

$$= \frac{\left(\frac{3\times12}{4}\right)^{th} + \left(\frac{3\times12}{4}+1\right)^{th}}{2} = \frac{9^{th}+10^{th}}{2}$$

$$Q_3 = \frac{31+39}{2} = 35$$

Inter-quartile range

$= Q_3 - Q_1 = 35 - 12.5 = 22.5$

$\therefore$ Inter-quartile range is 22.5

iv) **The Quartiles for grouped data**

The i^{th} position (rank) for the grouped data is determined by the formula used for ungrouped data. For grouped data, the cumulative frequency column is used to identify quartile class. The quartiles are calculated using the following formula:

$$Q_i = L_i + \left(\frac{\frac{i}{4}N - f_b}{f_w}\right)c \text{ , for; } i = 1,2,3$$

where:

L_i represents lower class boundary of the i^{th} quartile class.

f_b represents sum of the frequencies of all classes lower than the i^{th} quartile class (as per ascending order).

f_w represents the frequency of the i^{th} quartile class.

c represents class width.

N represents total frequencies (Observations)

i represents the position of the quartile

Example 6.23 Consider the frequency distribution Table 6.50

Table 6.50. Frequency distribution of grouped data

Class intervals	Frequencies
5 – 10	18
11 – 16	26
17 – 22	20
23 – 28	12
29 – 34	11
35 – 40	10
41 – 46	3

Calculate the following:

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a) the lower quartile Q_1

b) the upper quartile Q_3

Solution

a) To calculate Q_1

Table 6.51 Cumulative frequency distribution table.

Class intervals	Frequencies	Cumulative frequencies
5 – 10	18	18
11 – 16	26	44
17 – 22	20	64
23 – 28	12	76
29 – 34	11	87
35 – 40	10	97
41 – 46	3	100

The position of the $Q_1 = \left(\frac{N}{4}\right)^{th}$ position

$= \left(\frac{100}{4}\right)^{th}$ position

$= 25^{th}$ position

Thus, using the cumulative frequency, first quartile Q_1 is within the class interval 11-16. Thus;

$L_1 = 10.5,\ N = 100,\ f_w = 26,\ f_b = 18$

and $c = 6$

Using; $Q_1 = L_1 + \left(\frac{\frac{1}{4}N - f_b}{f_w}\right)c$

$= 10.5 + \left(\frac{\frac{100}{4} - 18}{26}\right) \times 6$

$= 10.5 + 1.62 = 12.12$

$\therefore$ The lower quartile is 12.12.

b) To calculate Q_3

The position of $Q_3 = \left(\frac{3}{4}N\right)^{th}$ position

$= \left(\frac{3}{4} \times 100\right)^{th}$ position

$= 75^{th}$

Thus using the cumulative frequency, the third quartile Q_3 is within the class interval of 23 – 28, and note that $L_3 = 22.5,\ N = 100,$

$f_w = 12,\ f_b = 64$ and $c = 6$

Using; $Q_3 = L_3 + \left(\frac{\frac{3}{4}N - f_b}{f_w}\right)c$

$= 22.5 + \left(\frac{\frac{3}{4} \times 100 - 64}{12}\right) \times 6$

$= 22.5 + \left(\frac{75 - 64}{12}\right) \times 6$

$= 2.5 + 5.5$

$= 8$

Therefore, the upper quartile Q_3 is 28.

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v) **Estimation of quartiles using cumulative frequency curve (Ogive)**

The three quartiles can also be estimated from the drawn cumulative frequency curve as follows:

1. The first quartile Q_1 is the corresponding value in the horizontal axis when the cumulative frequency is equivalent to the the position of $\frac{N}{4}$
2. The second quartile Q_2 or median is the corresponding value in the horizontal axis when the cumulative frequency is equivalent to $\frac{N}{2}$
3. The third quartile Q_3 is the corresponding value in the horizontal axis when the cumulative frequency is equivalent to $\frac{3}{4}N$ where N represent the total frequency.

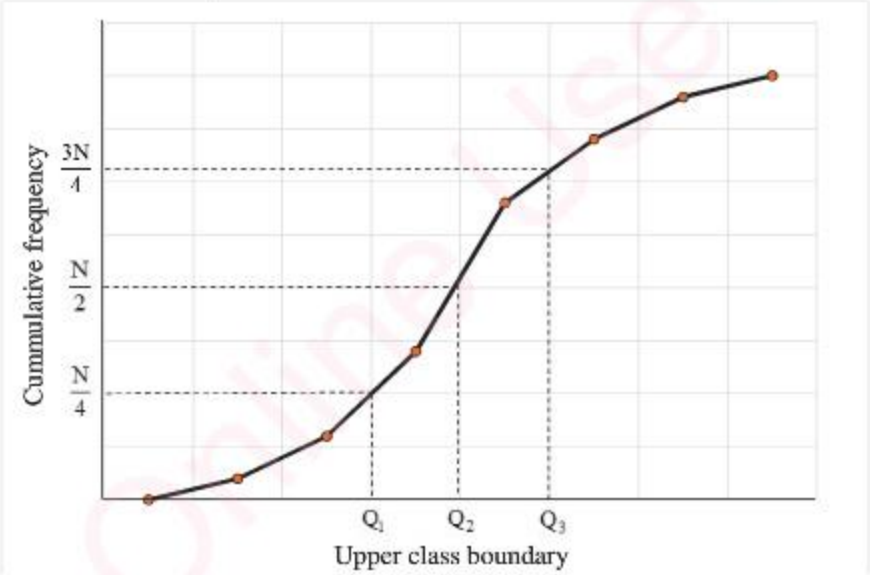

Figure 6.24 Illustration for estimating quartiles using cumulative frequency curve (ogive)

b) The Percentiles

In quartiles, the set of numbers are divided into four (4) equal parts. The knowledge of percentile relates to quartiles except that for percentiles the set of numbers is divided into one hundred (100) equal parts. The percentile are denoted by $P_1, P_2, P_3, \ldots, P_n$. Percentiles are position measures used in education and health fields to indicate the position of an individual in a group. It also provides information regarding how the data items are spread over the interval from the lowest to the highest value.

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Remarks:

1. The 25th percentile P_1 is equal to the lower quartile Q_1.
2. The 50th percentile P_{50} is equal to the second quartile Q_2, equal to the median.
3. The 75th percentile P_{75} is equal to the upper quartile Q_3.

i) Percentiles for ungrouped data

To compute the k^{th} percentile P_k, the locator or rank is identified using the following expression,

$$R_k = \left(\frac{k}{100}\right)N, \; k = 1,2,3,\ldots,99$$

Where

- k represent the k^{th} percentile
- N represents total number of data values (Total frequency).
- R_k represents the position (rank) of a given k^{th} percentile.

Example 6.24: Given $N = 120$ and $k = 55$. Find position (rank) of the given k^{th} percentile.

Solution

From $$R_k = \left(\frac{k}{100}\right)N$$

$$R_{55} = \left(\frac{55}{100}\right)\times 120 = 66^{th} \text{ position.}$$

Therefore, the 5th percentile is the 6th value starting from the lowest value of 120 values.

ii) Percentiles for grouped data:

The computation of percentile is similar to that of quartiles. To compute the k^{th} percentile P_k, the following formula is used:

$$P_k = L_k + \left(\frac{\frac{k}{100}N - f_b}{f_w}\right)c, \quad \text{where; } k = 1,2,3,\ldots,99.$$

where

- L_k represents the lower class boundary of the class containing k^{th} percentile.
- f_b represents the sum of frequencies before the k^{th} percentile class (lower classes).
- f_w represents frequency of the k^{th} percentile class.
- c represents the class size (width).

Example 6.25: Consider Table 6.52 which shows the performance of form two students in history subject for a certain school.

Table 6.52 Form two marks for History test.

Marks	Frequencies
10 – 19	1
20 – 29	8
30 – 39	12
40 – 49	20
50 – 59	18
60 – 69	10
70 – 79	7
80 – 89	4

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Calculate:

a) The 85^{th} percentile

b) The 50^{th} percentile

Solution

Prepare a cumulative frequency table

Table 6.53 Cumulative frequency table of History marks

Marks	Frequencies	Cumulative frequencies
10 – 19	1	1
20 – 29	8	9
30 – 39	12	21
40 – 49	20	41
50 – 59	18	59
60 – 69	10	69
70 – 79	7	76
80 – 89	4	80

Using; $R_k = \left(\frac{k}{100}\right)N$

$$R_{85} = \left(\frac{85}{100}\right)\times 80$$

$= 68^{th}$ data value (position).

Now, using the cumulative frequency column P_{85} is between 68^{th} and 69^{th} scores. Thus,

P_{85} class interval is $60 - 69$

$L_k = 59.5,\ f_w = 10,\ f_b = 59,\ c = 10$

$$P_k = L_k + \left(\frac{\frac{k}{100}N - f_b}{f_w}\right)c,$$

where; $k = 1,\ 2,\ 3,\ldots,\ 9$.

Hence, $P_{85} = 59.5 + \left(\frac{68-59}{10}\right)\times 10$

$= 59.5 + 9 = 68.5$

$\therefore P_{85} = 68.5$

a) Using; $R_k = \left(\frac{k}{100}\right)N$

$$R_{50} = \left(\frac{50}{100}\right)\times 80$$

$= 40^{th}$ data value.

Now, using cumulative frequency column P_{40} is between 40^{th} and 41^{th} scores thus,

P_{40} class interval is $40 - 49$

$L_k = 39.5,\ f_w = 20,\ f_b = 21,\ c = 10$

$$P_k = L_k + \left(\frac{\frac{k}{100}N - f_b}{f_w}\right)c,$$

where; $k = 1,\ 2,\ 3,\ldots,\ 9$.

Hence, $P_{50} = 39.5 + \left(\frac{40-21}{20}\right)\times 10$

$= 39.5 + 9.5 = 49$

$\therefore P_{50} = 49$

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Exercise 6.7

1. A year ago, Angela began working at a timber store. Her supervisor asked her to keep records of the number of sales she made each month. The following data set is a list of her sales for the last 12 months: 34, 47, 10, 15, 57, 24, 20, 11, 19, 50, 28, 37.

   Use Angela's sales records to find:

   a) the median.
   b) the upper and lower quartiles.
   c) the interquartile range.
   d) the semi interquartile range.

2. Find the median, lower quartile, upper quartile and inter quartile range of the following set of scores: 18, 20, 23, 20, 23, 27, 24, 23, 29

3. The heights of a sample of 100 students in a certain college were recorded as shown in Table 6.54.

   Table 6.54 Heights of students

Heights (Cm)	160 – 162	163 – 165	166 – 168	169 – 171	172 – 174
Frequencies	11	26	30	24	9

   Find:

   a) 70^{th} percentile
   b) 30^{th} percentile
   c) Semi-interquartile range

4. For each of the Tables 6.55 and 6.56, find P_{20}, P_{90}, Q_1 and Q_3

   a) **Table 6.55** Frequency distribution of grouped data

Class intervals	1 – 8	9 – 16	17 – 24	25 – 32	33 – 40
Frequencies	28	47	60	43	22

   b) **Table 6.56** Frequency of distribution of grouped data

Numbers	40 – 44	45 – 49	50 – 54	55 – 59	60 – 64	65 – 69
Frequencies	56	201	270	230	199	44

5. The marks of a Biology test for 60 students in a certain school were recorded as shown in Table 6.57.

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Table 6.57 Biology test results

Marks	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79
Frequencies	5	12	20	15	8

a) Draw a histogram and use it to estimate the mode

b) Draw a cumulative frequency curve (ogive) and use it to estimate the first quartile, the median and the third quartile.

6.3 Measures of dispersion for grouped and ungrouped data

Measures of central tendency (mean, median and mode) provide information on the data values at the centre of the data set. Measures of dispersion (range, variance and standard deviation) describe the pattern of deviation exhibited by the data. So, it is an indication of the spread of values around the centre of the distribution.

Remarks. Measures such as quartiles, deciles and percentiles are known as measure of position. However, sometimes are also referred as measure of central tendency to some cases as they are the same as median.

The Range

a) For ungrouped data

This is simply the difference between the highest and the lowest class values.

Example 6.26: Find the range of each of the following data sets:

a) 65, 66, 67, 68, 71, 73, 74, 77

b) 42, 54, 58, 62, 67, 77, 85, 93, 100

Solution

a) Range = 77 – 65 = 12

b) Range = 100 – 42 = 58

b) For grouped data

Range is obtained by taking the difference between upper class boundary of the highest interval and the lower class boundary of the lowest class interval.

Example 6.27: For the following frequency distribution table, calculate range.

Table 6.58 Frequency distribution table

Class intervals	Frequencies
21-25	4
26-30	9
31-35	16
36-40	10
41-45	10
46-50	1

Solution

Range = Upper class boundary of the highest class interval – lower class boundary of the lowest class interval.

Range = 50.5 – 20.5 = 30

The Variance

Variance is the sum of square of deviations divided by the number of values or can be defined as the average of squares of deviations. Deviation is the distance from the mean ($\bar{x}$) which indicates how far above or below the value x is from the mean. It is used to determine the consistency of a variable and spread of data.

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a) For ungrouped data

Let x_i represent variable with values $x_1, x_2, x_3, ..., x_n$ whose mean is $\bar{x}$

Now,

$$\text{Variance} = \frac{\text{Sum of squares of deviations}}{\text{Number of observations}}$$

$$= \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2+...+(x_n-\bar{x})^2}{N}$$

Hence, for ungrouped data,

$$\text{Var}(x) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{N} \text{ or}$$

$$\text{Var}(x) = \frac{\sum_{i=1}^{n}x_i^2}{N} - \left(\frac{\sum_{i=1}^{n}x_i}{N}\right)^2$$

where N represents total number of observations.

Other formulae for calculating variance are:

i) $\text{Var}(x) = \frac{1}{N}\sum_{i=1}^{n}d_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n}d_i\right)^2$

where $d_i = x_i - A$, which is known as assumed mean method.

ii) $\text{Var}(x) = c^2\left[\frac{1}{N}\sum_{i=1}^{n}u_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n}u_i\right)^2\right]$

where $u_i = \frac{d_i}{c} = \frac{x_i - A}{c}$, which is known as coding method.

b) For grouped data

Let x_i represent class marks for grouped data with values $x_1, x_2, x_3, ..., x_n$, whose mean is $\bar{x}$ and f_i are the frequencies. The formulae for computing variance are:

i) $\text{Var}(x) = \frac{\sum_{i=1}^{n}f_i(x_1-\bar{x})^2}{\sum_{i=1}^{n}f_i}$

or $\text{Var}(x) = \frac{\sum_{i=1}^{n}f_ix_i^2}{\sum_{i=1}^{n}f_i} - \left(\frac{\sum_{i=1}^{n}f_ix_i}{\sum_{i=1}^{n}f_i}\right)^2$

where $\sum_{i=1}^{n}f_i = N$

Remark: The formula,

$$\text{Var}(x) = \frac{\sum_{i=1}^{n}f_i(x_1-\bar{x})^2}{\sum_{i=1}^{n}f_i}$$

can also be used for ungrouped data with values $x_1, x_2, x_3, ..., x_n$ which occur with frequencies $f_1, f_2, f_3, ... f_n$, respectively.

ii) $\text{Var}(x) = \frac{1}{N}\sum_{i=1}^{n}f_id_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n}f_id_i\right)^2$

where, $d_i = x_i - A$, which is known as assumed mean method.

iii) $\text{Var}(x) = c^2\left[\frac{1}{N}\sum_{i=1}^{n}f_iu_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n}f_iu_i\right)^2\right]$

where, $u_i = \frac{d_i}{c} = \frac{x_i - A}{c}$, which is known as Coding method .

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Example 6.27. Given the data set: 32, 41, 47, 53, 57. Find the variance.

Solution

Using,

$$\bar{x}=\frac{\sum x}{N}=\frac{32+41+47+53+57}{5}$$

$$=\frac{230}{5}=46$$

Table 6.59 Table for variance

x	32	41	47	53	57
$(x_i-\bar{x})$	–14	–5	1	7	11
$(x_i-\bar{x})^2$	196	25	1	49	121

Now, $\text{Var}(x)=\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{N}$

$$\text{Var}(x)=\frac{196+25+1+49+121}{5}$$

$$=\frac{392}{5}=78$$

$\therefore \text{Var}(x)=78.4$

The Standard deviation

This is the useful measure of dispersion of all data based on the centre value (mean). The value of standard deviation portrays how the values are spread around the mean. The standard deviation is the measures of average deviation between all observed value in the sample and the sample mean. It is used to determine the spread out of data and its consistency. The standard deviation is calculated from the variance as the square root of variance. It is denoted by σ.

That is; Standard deviation$=\sqrt{\text{Variance}}$

For a variable x, $\sigma_x=\sqrt{\text{var}(x)}$

a) For ungrouped data

The formula for standard deviation for variable x is:

$$\text{Standard deviation}=\sqrt{\text{Variance}}$$

$$=\sqrt{\frac{\text{sum of squares of deviations}}{\text{number of observations}}}$$

$$\sigma_x=\sqrt{\frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+(x_3-\bar{x})^2+\ldots+(x_n-\bar{x})^2}{N}}$$

Hence, for ungrouped data,

$$\sigma_x=\sqrt{\text{var}(x)}=\sqrt{\frac{\sum_{i=1}^{n}(x_1-\bar{x})^2}{N}} \quad \text{or}$$

$$\sqrt{Var(x)}=\sqrt{\frac{\sum_{i=1}^{n}x_i^2}{N}-\left(\frac{\sum_{i=1}^{n}x_i}{N}\right)^2}$$

where, N represents total number of observations and $\bar{x}$ represents mean.

Other formulae for finding standard deviation are:

i) $\sigma_x=\sqrt{\left[\frac{1}{N}\sum_{i=1}^{n}d_i^2-\left(\frac{1}{N}\sum_{i=1}^{n}d_i\right)^2\right]}$

where, $d_i=x_i-A$, which is known as the assumed mean method.

ii) $\sigma_x=c\sqrt{\left[\frac{1}{N}\sum_{i=1}^{n}u_i^2-\left(\frac{1}{N}\sum_{i=1}^{n}u_i\right)^2\right]}$

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where $u_i = \frac{d_i}{c} = \frac{x_i - A}{c}$, which is known as coding method.

b) For grouped data

Let x_i represent class marks with values $x_1, x_2, x_3, \ldots, x_n$, $x_1, x_2, x_3, \ldots, x_n$ whose mean is $(\bar{x})$ and f_i are the frequencies. Formulae for computing standard deviation are:

i) $$\sigma_x = \sqrt{\text{var}(x)} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 f_i}{N}} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 f_i}{\sum_{i=1}^{n} f_i}}$$

Remark: The formula $\sigma_x = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 f_i}{\sum_{i=1}^{n} f_i}}$ can also be used for ungrouped data values $x_1, x_2, x_3, \ldots, x_n$ which occur with frequencies $f_1, f_2, f_3, \ldots, f_n$ respectively.

ii) $$\sigma_x = \sqrt{\left[\frac{1}{N}\sum_{i=1}^{n} f_i d_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)^2\right]}$$

where, $d_i = x_i - A$, which is known as ***Assumed Mean Method.***

iii) $$\sigma_x = \sqrt{\left[\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2\right]}$$

where, $u_i = \frac{d_i}{c} = \frac{x_i - A}{c}$, , which is known as ***Coding method***.

Example 6.28: Consider Table 6.60

Table 6.60 Grouped data values

Class intervals	Frequencies
60 – 62	5
63 – 65	18
66 – 68	42
69 – 71	27
72 – 74	8

Calculate;

a) the Variance
b) the Standard deviation

Solution

a) Using;

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{6745}{100} = 67.45$$

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Table 6.61 Table for solution of Example 6.28

Class Intervals	f_i	x_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x - \bar{x})^2$
60 – 62	5	61	305	–6.45	41.6025	208.0125
63 – 65	18	64	1152	–3.45	1.9025	214.245
66 – 68	42	67	2814	–0.45	0.2025	8.505
69 – 71	27	70	1890	2.55	6.5025	175.5675
72 – 74	8	73	584	5.55	30.8025	246.42
	$\sum_{i=1}^{n} f_i = 100$		$\sum_{i=1}^{n} f_i x_i = 6745$ $\bar{x} = 67.45$			$\sum_{i=1}^{n} (x_i - \bar{x})^2 f_i$ $= 852.75$

Using $Var(x) = \dfrac{\sum_{i=1}^{n} (x_i - \bar{x})^2 f_i}{\sum_{i=1}^{n} f_i}$

$$= \frac{852.75}{100}$$

$$= 8.5275$$

$\therefore \text{Var}(x) = 8.5275$

b) Using $\sigma_x = \sqrt{\dfrac{\sum_{i=1}^{n} (x_i - \bar{x})^2 f_i}{\sum_{i=1}^{n} f_i}}$

$$\sigma_x = \sqrt{\text{Var}(x)}$$

$$\sigma_x = \sqrt{8.5275}$$

$$= 2.9202$$

Standard deviation is 2.9202

Alternatively:

By using assumed mean method and coding method:

Let assumed mean, $A = 67$

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Table 6.62 Table for alternative of Example 6.28

Class intervals	f_i	x_i	$d_i = x_i - A$	$f_i d_i$	$f_i d_i^2$	$u_i = \frac{x_i - A}{c}$	$f_i u_i$	$f_i u_i^2$
60 – 62	5	61	–6	–30	180	–2	–10	20
63 – 65	18	64	–3	–54	162	–1	–18	18
66 – 68	42	67	0	0	0	0	0	0
69 – 71	27	70	3	81	243	1	27	27
72 – 74	8	73	6	48	288	2	16	32
	$\sum_{i=1}^{5} f_i = 100$			$\sum_{i=1}^{5} f_i d_i = 45$	$\sum_{i=1}^{5} f_i d_i^2 = 873$		$\sum_{i=1}^{5} f_i u_i = 15$	$\sum_{i=1}^{5} f_i u_i^2 = 97$

a) i) $\text{Var}(x) = \frac{1}{N}\sum_{i=1}^{n} f_i d_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)^2$

$= \frac{1}{100}(873) - \left(\frac{45}{100}\right)^2 = 8.5275$

ii) $\text{Var}(x) = c^2\left[\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2\right]$

$= 3^2\left[\frac{97}{100} - \left(\frac{15}{100}\right)^2\right]$

$= 8.5275$

b) i) $\sigma_x = \sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i d_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i d_i\right)^2}$

$= \sqrt{\frac{1}{100}(873) - \left(\frac{45}{100}\right)^2} = 2.9202$

ii) $(\sigma_x) = c\sqrt{\frac{1}{N}\sum_{i=1}^{n} f_i u_i^2 - \left(\frac{1}{N}\sum_{i=1}^{n} f_i u_i\right)^2}$

$= 3\sqrt{\frac{97}{100} - \left(\frac{15}{100}\right)^2} = 2.9202$

Exercise 6.8

1) Find the range for each set of the following data:
   a) 2, 11, 20, 3, 19, 50, 40, 33, 26
   b) 11, 21, 31, 41, 51, 61, 71, 81, 91
2) Find the variance and the standard deviation for each of the following sets of data.
   a) 1, 2, 3, 4, 5, 6, 7, 8, 9
   b) 12, 18, 20, 21, 29

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3) Calculate the variance and standard deviation in Table 6.63 and Table 6.64.

a)

Table 6.63 Table of ungrouped data

Numbers	11	12	13	14	15	16
Frequencies	14	21	22	19	18	6

b)

Table 6.64 Table of grouped data

Masses	50 – 52	53 – 55	56 – 58	59 – 61	62 – 64
Frequencies	7	20	41	29	3

4) Use the following three data set A, B and C to answer the questions that follows:

$A = \{9, 10, 11, 7, 13\}$

$B = \{10, 10, 10, 10, 10\}$

$C = \{1, 1, 10, 19, 19\}$

a) Find the mean for each data set.

b) Find the standard deviation of each data set.

c) Which data set has the largest standard deviation?

d) Is it possible to answer question 4(c) above without calculating the standard deviation? Explain.

5) A quality control section of a cannery inspected the content of 100 randomly selected tins of cooked spaghetti from output. As part of the measurements, Table 6.65 displays the net weights (in grams) obtained.

Table 6.65 Weight of spaghetti

Weights (in grams)	33 – 36	37 – 40	41 – 44	45 – 48	49 – 52
Number of tins	15	17	21	22	25

Calculate the mean net weight of the contents of the tins and standard deviation.

6) For a set of ungrouped data values, the following sums are found:

$$n = 15, \sum x_i = 480, \sum x_i^2 = 15435.264$$

Find the standard deviation.

7) The goals scored by two teams A and B in a competition match are presented in Table 6.66.

Table 6.66 Match results for teams A and B

No. of goals scored in a match		0	1	2	3	4
No. of matches	A	27	9	8	5	4
	B	17	9	6	5	3

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Use the assumed mean of 2 for both teams to compute the means and standard deviations for team A and B .

6.4 Applications of statistics

Generally, statistics is applied in every sector. It is used in education, business, economics, researches, sports and games, projects, health, and agriculture, just to mention a few. The information from these fields can be computed using statistics for scientific decision making. The following are some of the areas where statistics is applied.

Government agencies

The government uses statistics to make decisions about populations, health, education and in other sectors. Statistics is used in education to analyse findings of research on strengths and weaknesses of curriculum implementation. The national population census apply statistics to analyse variables assessed in the survey.

Science and Medicine

The medical field would be far less effective without research to see which medications or interventions work best. Medical professionals also perform studies by race, age, or nationality to see how they react to different medications.

Education

Teachers are encouraged to do researches and apply statistical packages in order to evaluate relevant and effective teaching and learning process. Also to evaluate test items to determine the performance of students. In this aspect, the statistical techniques play a vital role in data analysis to inform educational stakeholders with relevant information for decision making.

Activity 6.3

In groups, identify at least two areas such as in education, business, medicine or any other field of your choice where statistics is applied. Collect and organise your data into classes using frequency distribution table. Use appropriate statistical approaches such as mean, median, mode, standard deviation, variance, graphs or any other of your choice, analyse the data you have collected. Use the findings to prepare a short report with key recommendations. Present your final work to the class and your teacher for further discussion.

Revision exercise 6

1. Using the coding method, find the mean of 88 students' marks obtained in a test as given in Table 6.67.

Table 6.67 Students marks

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
Frequencies	6	16	24	25	17

2. Table 6.68 shows the duration of service in years by employees who have been working for a certain company for at least ten years.

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Table 6.68 Employees years of service

Duration of service	10 – 15	15 – 20	20 – 25	25 – 30	30 – 35	35 – 40	40 – 45
Frequencies	30	42	20	23	13	8	4

Calculate: a) The mean and standard deviation of the length of service of these employees by using coding method.

b) The median and mode

3. Table 6.69 gives the frequency distribution of the number of orders received each day for the past 50 days at the office of Mail-Order Company. Find the mean and the quartiles (Q_1, Q_2, and Q_3). What can you conclude about Q_2?

Table 6.69 The order by Mail Order Company

Number of orders	Frequencies
10 – 12	4
13 – 15	12
16 – 18	20
19 – 21	14

4. Use the grouped data in Table 6.70:

Table 6.70 Grouped data

Time to travel to work	Frequencies
1 – 10	8
11 – 20	14
21 – 30	12
31 – 40	9
41 – 50	7

Find

a) the quartiles of the data
b) the inter-quartile range of the data

5. Find the mean age, standard deviation, and the 25th and 75th percentiles from the data in Table 6.71.

Table 6.71 Table of grouped data

Ages in years	1 – 10	11 – 20	21 – 30	31 – 40
Number of individuals	34	18	17	11

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6. Given grouped data in Table 6.72;

Table 6.72 Grouped data

Class intervals	Frequencies
11 – 15	2
16 – 20	5
21 – 25	4
26 – 30	3
31 – 35	6
36 – 40	5

a) Use the cumulative frequency curve to estimate Q_1 and Q_3

b) Use histogram to estimate the mode

7. The marks obtained by 88 students in Biology examination were recorded as shown in Table 6.73.

Table 6.73 Biology examination results

Marks	1 – 11	11 – 21	21 – 31	31 – 41	41 – 51
Frequencies	6	16	24	25	17

Find the mean, median and mode.

8. Table 6.74 is the frequency distribution of marks obtained by 60 students in a test.

Table 6.74 Marks for 60 students

Marks	0	1	2	3	4	5
Frequencies	$x-2$	x	x^2	$(x-1)^2$	$2x$	$x+1$

Determine the mean and the standard deviation of the marks.

9. The weights of coffee in 70 jars are shown in Table 6.75.

Table 6.75 Weight of jars of coffee

Weights (in grams)	Frequencies
200 – 201	13
201 – 202	27
202 – 203	18
203 – 204	10
204 – 205	1
205 – 206	1

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Determine:

a) Variance and standard deviation
b) The median and mode

10. Prepare the frequency distribution table for the following marks. Use the intervals 1 – 10, 11 – 20,...

25	31	28	11	18	12	15
37	42	48	34	13	15	32
16	34	14	34	28	41	17

From the distribution table, find:

a) The mean by using coding method
b) The quartiles
c) The semi inter-quartile range
d) The 35th percentile.

11. Table 6.76 shows the frequency distribution of the age of employees in a certain company

Table 6.76 Age of employees

Ages	20 – 25	25 – 30	30 – 35	35 – 40	40 – 45	45 – 50
No. of employees	170	110	80	45	40	35

Calculate the standard deviation of the age of the employees.

12. Table 6.77 gives the number of finished bottles turned out per day by different number of workers in a factory. Find the standard deviation of the daily output of finished bottles.

Table 6.77 Finished bottles

Number of bottles	18	19	20	21	22	23	24	25	26	27
Number of workers	3	7	11	14	18	17	13	8	5	4

13. Use a computer spreadsheet package to draw the histogram and bar graph by using the data given in Table 6.78.

Table 6.78 Grouped data

Heights (cm)	12	17	22	27	32	37	42	47	52	57	62	67	72
Frequencies	8	10	15	25	30	40	38	16	22	14	10	5	2

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Chapter Seven

Probability

Introduction

Probability is about prediction of uncertainties which require the law of chance. It is used in science, technology, commerce, economics, social science, and sports to predict the occurrence of some events. In this chapter, you will describe and use permutations and combinations in finding probability sample space, calculate probability of an event, combined events, mutual exclusive events, independent events, and apply probability in real-life.

7.1 Permutations and Combinations

Ways of arranging 'n' unlike objects

If one item can be selected in n different ways and another item can be selected in m different ways; then, both items taken together may be selected $n \times m$ different ways. The method of determining $n \times m$ ways of selecting an item to determine the number of ways both items can be selected is called the fundamental principle of counting.

Example 7.1: Juma has four different coloured shirts and six other different coloured soccer jerseys from which he can choose an outfit for a final football match. How many different outfits can he arrange for himself?

Solution

For each of the four coloured shirts, Juma has a choice of six soccer jerseys, so all together he has $4 \times 6 = 24$ different outfits.

Hence, in this way of selecting objects someone may use with repetition or without repetition. In the fundamental principle of counting, one may solve problems by using either a tree diagram or a table. Suppose three letters A, B, and C which are to be arranged in a line. If repetition of letters is not allowed, then the number of ways can be found as shown in Figure 7.1.

a) By using a tree diagram

i) Without repetition

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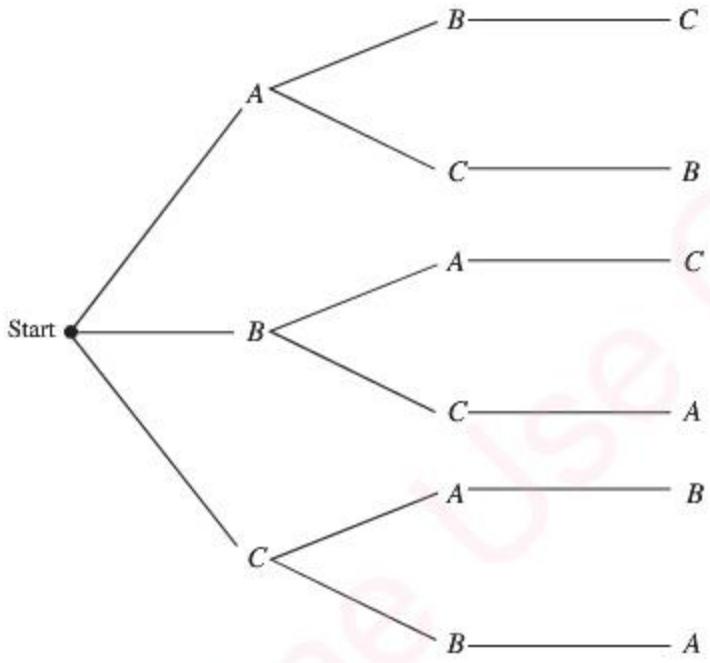

Figure 7.1 A tree diagram showing the arrangements of three letters without repetition

From Figure 7.1, the possible arrangement is

{*ABC, ACB, BAC, BCA, CAB, CBA*} which is equal to six possible number of arrangements.

ii) With repetition

If repetition of letters is allowed, then the number of ways can be found as shown in Figure 7.2.

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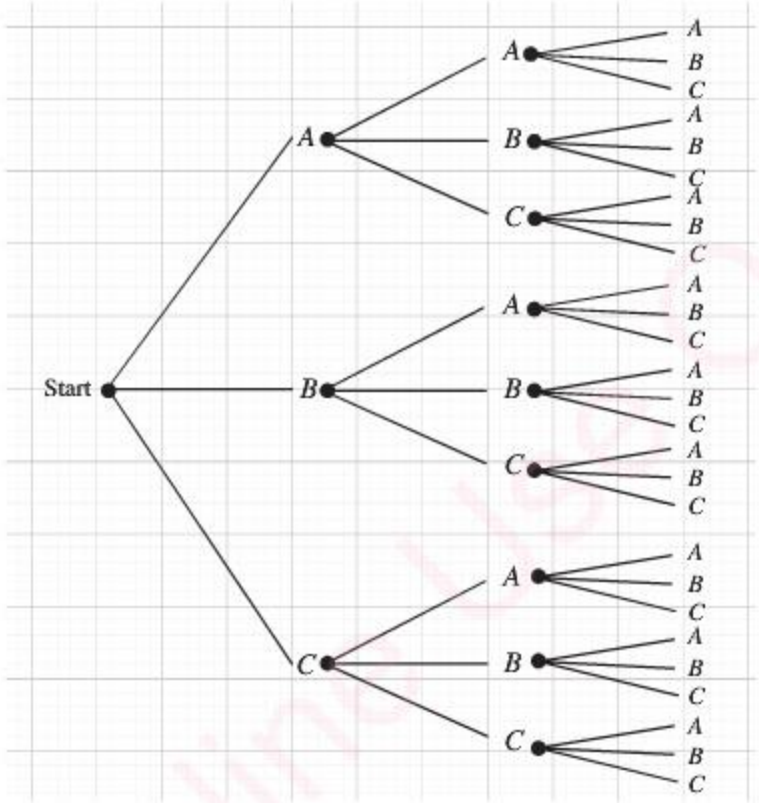

Figure 7.2 The tree diagram showing the arrangement of three letters with repetition

From Figure 7.2, the possible arrangement is:

$\{AAA, AAB, AAC, ABA, ABB, ABC, ACA, ACB, ACC, BAA, BAB, BAC, BBA, BBB, BBC, BCA, BCB, BCC, CAA, CAB, CAC, CBA, CBB, CBC, CCA, CCB, CCC\}$ which is equal to 27 ways.

b) By using a table

The ways of arrangement can be obtained by using a table.

i) If repetition of letters is not allowed, the number of ways can be found as it is summarized in Table 7.1.

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Table 7.1 Arrangement of letters without repetition

Position of location	1st choice	2nd choice	3rd choice
Number of choices	3	2	1

Now, the total number of ways $= 3\times 2\times 1$

$= 6$ ways.

ii) If repetition of letters is allowed, then the number of ways can be found as it is summarized in Table 7.2:

Table 7.2 Arrangement of letters with repetition

Position of location	1st choice	2nd choice	3rd choice
Number of choices	3	3	3

Now, the total number of ways $3\times 3\times 3 = 27$ Ways

Factorial notation

If n is a natural number, then $n!$ is read as n-factorial and can be expressed as follows:

$n! = n(n-1)(n-2)(n-3)....(3)(2)(1)$

Example 7.2: Evaluate a) 5! b) 9! c) $\frac{4!}{2!3!}$

Solution

a) $5! = 5\times(5-1)\times(5-2)\times(5-3)\times(5-4)$

$= 5\times 4\times 3\times 2\times 1$

$\therefore 5! = 120$

b) $9! = 9\times(9-1)\times(9-2)\times(9-3)\times(9-5)\times(9-6)\times(9-7)\times(9-8)$

$9! = 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1$

$\therefore 9! = 362\,880$

c) $\frac{4!}{2!3!} = \frac{4\times 3}{2!\times 3!}$

$= \frac{4}{2\times 1}$

$\therefore \frac{4!}{2!3!} = 2$

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Example 7.3 A business owner has a choice of eight locations to establish business. The locations can be ranked according to certain criteria, such as price of the store and parking facilities.

a) How many different ways can eight locations be ranked?

b) How many different ways can the top four locations be ranked?

Solution:

a) There are $8! = 8\times7\times6\times5\times4\times3\times2\times1 = 40{,}320$ different possible rankings.

b)

Position of locations	First choice	Second choice	Third choice	Fourth choice
Number of choices	8	7	6	5

Using the fundamental counting rule, the total different possible rankings

$= 8\times7\times6\times5 = 1{,}680.$

Therefore the top four locations can be ranked into 1,680 ways.

Exercise 7.1

1. How many words can be formed using all letters of the words
   a) MAGU
   b) TEMBO
   c) HISTORY
2. How many four digit numbers can be formed using the digits 1, 2, 3, and 4 if:
   a) repetition is allowed
   b) repetition is not allowed
3. Simplify the following factorial expressions.
   a) $\frac{20!5!}{18!4!}$
   b) $\frac{7!-5!}{6!-4!}$
4. In how many ways can seven children be arranged in a row so that two of them are always
   a) sitting together?
   b) not sitting together?

Permutations

When an individual wants to arrange certain different objects in different ways, if these objects are arranged to follow a certain order, that arrangement is called permutation. Thus, permutation may refer to an arrangement of the number of objects in a particular order.

Remark: The order of arrangement of objects is the key concern in permutation.

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Permutation of n objects taking r at a time

If P denotes the number of permutations (arrangements) of n different objects when r objects are taken at a time and $r<n$, then the formula for permutation is given by $^nP_r=\frac{n!}{(n-r)!}$

Example 7.4: Evaluate the following:

a) $^5P_3$ b) $^{10}P_8$

Solution

a) $^5P_3=\frac{5!}{(5-3)!}=\frac{5\times4\times3\times2!}{2!}$

$\therefore {}^5P_3=60$

b) $^{10}P_8=\frac{10!}{(10-8)!}=$

$=\frac{10!}{2!}=\frac{10\times9\times8\times7\times6\times5\times4\times3\times2!}{2!}$

$^{10}P_3=1,814,400$

Example 7.5: In how many different ways can 8 students be seated on 6 chairs?

Solution

Given, $n=8, r=6$

Now $^nP_r=\frac{n!}{(n-r)!}$

$\frac{8!}{2!}=\frac{8\times7\times6\times5\times4\times3\times2\times1}{2!}$

20,160

Therefore, students can be seated in 20,160 different ways.

Also, permutation (arrangement) of 'n' different objects when r objects are taken at a time and $r=n$, is given as follows:

From $^nP_n=\frac{n!}{(n-r)!}$

Given $n=r$

$^nP_n=\frac{n!}{(n-r)!}$

$^nP_n=\frac{n!}{0!}$

Recall $0!=1$

$\therefore {}^nP_n=n!$

Example 7.6: Halima has 5 books which are different in colour and size. She wants to arrange those books by taking 5 at a time. How many ways can she arrange the books?

Solution

Given $n=5$, and $r=5$

From, $^nP_r=\frac{n!}{(n-r)!}$, since $r=n$,

then, $^nP_n=n!$

Thus $^5P_5=5!=120$

Therefore, Halima can arrange the books in 120 different ways.

Also, if some of the objects to be arranged are alike and their number is r, and if q is another number of alike objects and n is a total number of objects, then the number of permutations is given by: $^nP_n=\frac{n!}{r!q!}$

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Example 7.7: How many eight letter patterns can be formed from the letters in the name "MOROGORO"?

Solution

From the name "MOROGORO", we have four "O" letters which are alike and two "R" letters which are alike.

Now:

From $P=\frac{n!}{r!q!}$

Given $n=8,\ r=4$ and $q=2$

Then, $P=\frac{8!}{4!2!}$

$=\frac{40320}{48}$

$=840$

Thus, there are 840 different eight-letter patterns.

Remark: The value of 1! is 1. It can be proved as follows: $1!=0!=1$

Let "n" be any natural number.

Then, $n!=n(n-1)!$

Divide by n both sides,

$\frac{n!}{n}=(n-1)!$

Setting $n=1$

Then, $\frac{1!}{1}=(1-1)!$

$\therefore\ 1!=0!$ shown

Exercise 7.2

1. Evaluate each of the following:
   a) ${}^{10}P_1$ b) ${}^{4}P_3$ c) ${}^{11}P_9$ d) ${}^{25}P_{23}$
2. How many ways can all the letters in the following words be arranged without repetition.
   a) ANTANANARIVO
   b) MADINI
   c) BENZENE
   d) MAKINIKIA
3. In how many ways can the product r^2s^3t be written without using exponents?
4. A box contains one red ball, one white ball, one yellow ball, and one black ball. In how many different ways can three of the balls be arranged?
5. There are three routes joining town A and B and five routes joining B and C. Find the number of different ways of moving from A to C via town B.
6. There are 13 Tanzanians, 10 Kenyans, and 4 Ugandans in the class. In how many different ways can a Tanzanian, a Kenyan, and Ugandan be arranged in the class?
7. In how many ways can 9 students be seated in a row of 6 chairs?
8. How many five digits can be formed using the digits 1, 2, 3, 6, and 9 without repetition?
9. How many four letter words can be formed using letters in the word VALUES without repetition such that the last letter is a vowel?
10. How many different ways can letters of the word TANZANITE be arranged without repetition?

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Combinations

A combination is the selection of distinct objects without regarding the order.

Combination of n objects taking r objects at a time

If ${}^nC_r$ denotes the number of combinations on selection of n different objects when r objects are taken at a time, where $r \leq n$

then, $${}^nC_r = \frac{n!}{(n-r)!r!}.$$

The following are relations of the combination:

a) ${}^nC_r = {}^nC_{n-r}$

b) ${}^nC_n = 1$

c) ${}^nC_0 = 1$

d) $r!({}^nC_r) = {}^nP_r$

Example 7.8: How many committees of 4 people can be selected from a group of 10 people?

Solution

Given $n = 10$, $r = 4$.

Then $${}^{10}C_4 = \frac{10!}{(10-4)!4!}$$

$$= \frac{10!}{6!4!}$$

$$= \frac{10\times9\times8\times7\times6!}{6!4!}$$

$$= \frac{10\times9\times8\times7}{4\times3\times2\times1}$$

$$= 210$$

$\therefore$ 210 committees can be selected.

Example 7.9: How many selections of 4 letters can be made from the 6 letters a, b, c, d, e and f?

Solution

The number of selections is ${}^nC_r$ where r is the number of things selected from a group of n objects.

Hence, for this case $n = 6$ and $r = 4$, thus ${}^6C_4 = \frac{6!}{2!4!} = 15$. Therefore, there are 15 selections of 4 letters which can be made from the 6 letters.

Activity 7.1

List the number of ways to colour the diagram below under the following two conditions:

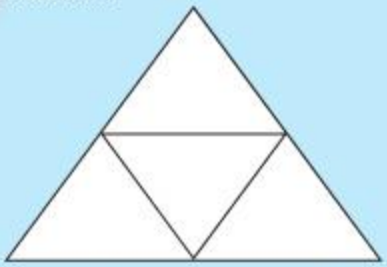

Figure 7.3 Triangles to be printed with different colours

a) Each of the smaller triangles is to be painted with one of three colours: red, blue or green.

b) No two adjacent regions have the same colour.

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Exercise 7.3

1. A committee of seven members is to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of exactly 3 girls?
2. From a committee of 6 people, how many ways can we choose a chairperson and a vice chairperson, assuming that one person cannot hold more than one position?
3. From a committee of 8 people, how many ways can we choose a chairperson, vice-chairperson, and secretary; assuming one person cannot hold more than one position?
4. Find the number of combinations of 7 objects taken 4 at a time.
5. Find the number of combinations of 12 objects taken 3 at a time.
6. There are 11 members in a team. If each member is to shake hand with every other member once, how many handshakes will be observed?

7.2 Probability of an event

The probability of an event is the number of outcomes an event can occur divided by the total number of possible outcomes. A set of all possible outcomes is called a sample space (S) while a specified outcome is called an event (E). The probability of an event is a ratio that measures how likely an event is to happen. The probability of an event is denoted by $P(E)$ and it is read as "the probability that event (E) occurs". It is given by:

$$PE = \frac{\text{Number of ways event } E \text{ can occur}}{\text{Total number of possible outcomes}}$$

$$P(E) = \frac{n(E)}{n(S)}$$

Probabilities are always numbers between 0 and 1 $(0 \le P(E) \le 1)$ and can be expressed as fractions, decimals, or percentages. The closer the probability of an event to 0, the less likely the event is to occur. An event with the probability of 0 can never happen. The closer the probability of an event to 1; the more likely is the event to occur. An event with a probability of 1 is certain to happen.

Example 7.10: A die is rolled once, what is the probability of getting,

a) an even number
b) a number greater than 4

Solution

Consider a die in Figure 7.4.

Figure 7.4 A die

Given that sample space after rolling a die is:

$$(S) = \{1,2,3,4,5,6\},\ n(S) = 6$$

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(a) Let A represents an even number that occurs

$$A=\{2,4,6\}$$

$$\therefore\ n(A)=3$$

$$\text{Now,}\quad P(A)=\frac{n(A)}{n(S)}$$

$$=\frac{3}{6}=\frac{1}{2}$$

$\therefore$ The probability of an even number to occur is $\frac{1}{2}$.

b) Let E represents a number greater than 4 that occurs

$$E=\{5,6\}$$

$$\therefore\ n(E)=\{2\}$$

$$\text{Now, } P(A)=\frac{n(E)}{n(S)}$$

$$P(A)=\frac{2}{6}=\frac{1}{3}$$

Therefore, the probability of a number greater than 4 to occur is $\frac{1}{3}$

The Tsh 100 coin has two sides, one for head and another for antelopes. The same applies for Tsh 200 coin, has head and lions.

For Tsh 500, H represents head and T represent animal side. When tossing a coin the expected outcomes are either;

i. getting head up, H
ii. getting an animal up, T

In short $S=\{H,T\}$. When British penny is tossed, it gives the same result as $\{H,T\}$. In this case, tossing a coin has two expected outcomes of Head (H) or a Tail (T).

Example 7.11: A coin is tossed twice. What is the probability of getting two tails?

Solution

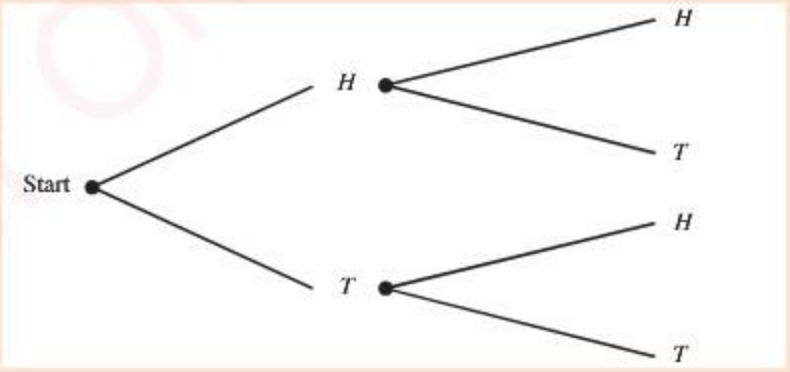

Figure 7.5 Tree diagram

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From Figure 7.5, sample space = {HH, HT, TH, TT} and event = {TT}

$$\text{Probability of getting two tails} = \frac{\text{number of two tails}}{\text{number of sample space}}$$

$$P(T) = \frac{n(T)}{n(S)} = \frac{1}{4}$$

Activity 7.2

A game involves rolling two dice together. The game has the following rules to win:

1. Win if both dice shows an even number.
2. Win if at least one of the numbers on top is even
3. Win if only one of the numbers appeared is even

Which set of the rules would you prefer?

Activity 7.3

List nine (9) events of your choice in Table 7.3. Put a tick in the column which best describes the likely-hood of each event. The first event is done as an example.

Table 7.3 Events of your choice

S/N	Event	Impossible	Very unlikely	Unlikely	50% chance	Likely	Very likely	Certain
1.	Alphonse driving his car from Tanga to Kigoma in two hours	√						
2.								
3.								
4.								
5.								
6.								
7.								
8.								
9.								

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Activity 7.4

A manufacturer has five seemingly identical computer terminals available for shipping. It is known that two of the five are defective. A particular order calls for two of the terminals and is filled by randomly selecting two of the five that are available. List the sample space for this experiment.

a) Let A denote the event that the order is filled with two non-defective terminals. List the sample points in A.

b) Assign probabilities to the sample events in such a way that the information about the experiment is used.

c) Find the probability of an event A.

Exercise 7.4

1. Two letters are randomly chosen, one after another, from the word TACK. List the sample space.
2. Suppose you plan to make a survey of families having two children. You want to record the sex of each child in the order of their births. For example, if the first child is a boy and the second a girl, you record $\{(\text{boy,girl})\}$. This is one point in the sample space. List all the sample points.
3. What is the probability of choosing an even number from $\{1,2,3,4,...,10\}$?
4. An integer between 10 and 20 inclusive is chosen at random. What is the probability that the integer chosen is: a) odd b) even
5. A marble is drawn randomly from a bag containing ten identical marbles, four of which are red and six are blue. Find the probability that the marble drawn is blue.

7.3 Combined events

Two or more events represented by a single event are called combined events. Finding probability of combined events, requires the use of tables, tree diagrams or probability laws.

a) Tree diagram of combined events

A Tree diagrams is one of the ways of organizing outcomes in order to identify probabilities. For example, if a box contains two red, two green and two white balls, and if two balls are chosen randomly, then the tree diagram (Figure 7.6) can be used to identify the possible combinations of the outcomes.

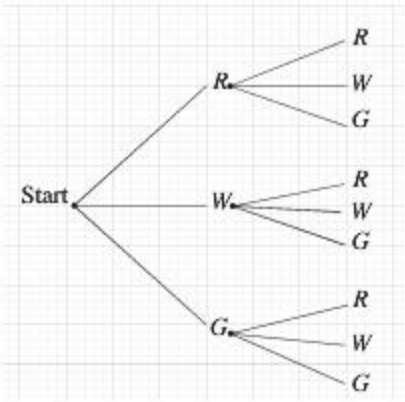

Figure 7.6 Tree diagram

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From the tree diagram, there are nine possible outcomes.

Hence, sample space $= \{RR, RW, RG, WR, WW, WG, GR, GW, GG\}; n(S) = 9$

b) A table showing combined events:

A table can help in finding some combined events. Suppose two coins are tossed at once, then one could arrange the results as follows:

Table 7.4 Combined events

2nd Coin ↓ / 1st Coin →	H	T
H	HH	HT
T	TH	TT

Hence, sample space $S = \{HH, HT, TH, TT\}$

$\therefore n(S) = 4$

Example 7.12: Three coins are tossed at once. Find the probability of obtaining:

a) three tails

b) one head and two tails

c) at least two heads

Solution

Let the head of the coin represented by (H) and tail represented by (T) sides. Hence, using the tree diagram, the arrangement is as follows:

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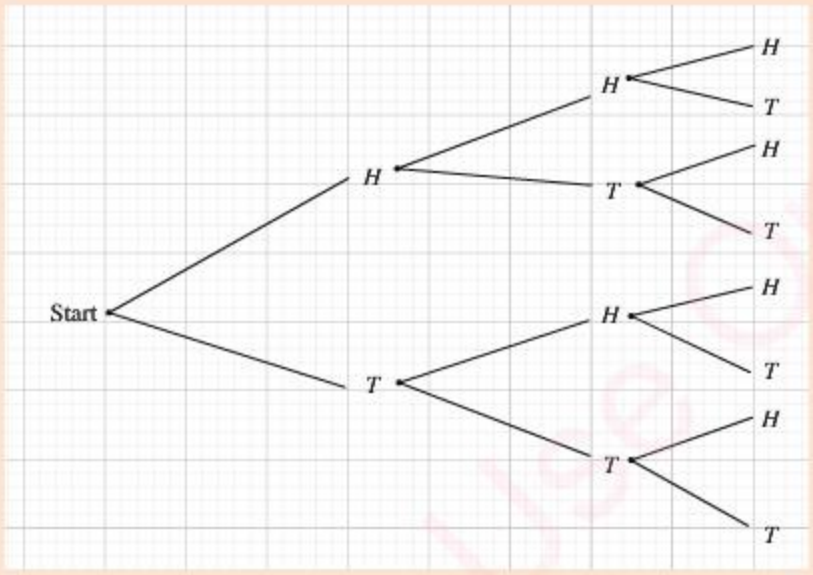

Figure 7.7 Tree diagram for three coins

Sample space,

$$S=\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$n(S)=8$$

a) Let event A represent three tails

$$A=\{TTT\}$$

$$n(A)=1$$

$$P(A)=\frac{n(A)}{n(S)}=\frac{1}{8}$$

$$\therefore P(A)=\frac{1}{8}$$

b) Let event B represent one head and two tails

$$B=\{HTT,THT,TTH\}$$

$$n(B)=3$$

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$$P(B)=\frac{n(B)}{n(S)}=\frac{3}{8}$$

$$\therefore P(B)=\frac{3}{8}$$

c) Let event C represent at least two heads

$$C=\{\text{HHH, HHT, HTH, THH}\}$$

$$n(C)=4$$

$$P(C)=\frac{n(C)}{n(S)}=\frac{4}{8}$$

$$=\frac{4}{8}$$

$$\therefore P(C)=\frac{1}{2}$$

c) The probability of combined events using a formula

For any combined events A and B from a sample space S,

$$P(A\cup B)=P(A)+P(B)-P(A\cup B)$$

Example 7.13: If $P(A)=\frac{1}{4}, P(B)=\frac{1}{3}$ and $P(A\cup B)=\frac{1}{2}$, find $P(A\cap B)$.

Solution

From $P(A\cup B)=P(A)+P(B)-P(A\cup B)$

Making $P(A\cap B)$ as the subject, we obtain

$P(A\cap B)=P(A)+P(B)-P(A\cup B)$

Then;

$$P(A\cap B)=\frac{1}{4}+\frac{1}{3}-\frac{1}{2}$$

$$=\frac{3+4-6}{12}$$

$$\therefore P(A\cap B)=\frac{1}{12}$$

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Example 7.14: If a six sides die is rolled, let B be an event in which the number that shows up is even, and let C be an event in which the number is greater than 4. Find the probability that the number that shows up is even or greater than four.

Solution

Given $S=\{1,2,3,4,5,6\}; n(S)=6$

$B=\{2,4,6\}; n(B)=3$

$P(B)=\frac{3}{6}=\frac{1}{2}$

$C=\{5,6\}; n(C)=2,\ P(C)=\frac{2}{6}=\frac{1}{3}$

$B\cap C=\{6\}; n(B\cap C)=1;\ P(B\cap C)=\frac{1}{6}$

$P(B\cup C)=P(B)+P(C)-P(B\cap C)$

$=\frac{3}{6}+\frac{2}{6}-\frac{1}{6}$

$=\frac{4}{6}=\frac{2}{3}$

Therefore, the probability that the number is even or greater than 4 is $\frac{2}{3}$.

Example 7.15: Given that $P(A)=0.5$, $P(B)=0.7$, and $P(A\cap B)=0.3$. By using Venn diagram, find:

(a) $P(A'\cup B')$ (b) $P(B\cap A')$ (c) $P(A\cup B')$

Solution

(a) Consider the Figure 7.8

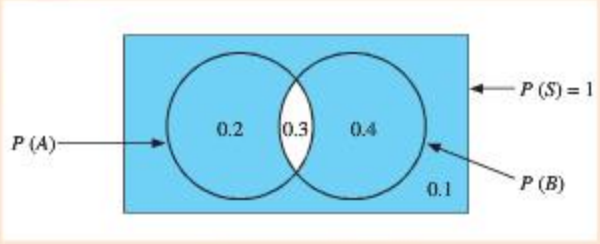

Figure 7.8 Venn diagram 1

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The shaded region represents $P(A' \cup B')$

$$P(A' \cup B') = 1 - P(A \cap B)$$
$$= 1 - 0.3$$
$$= 0.7$$
$$P(A' \cup B') = 0.7$$

Alternatively

$$P(A' \cup B') = \text{Sum of probabilities in the shaded region}$$
$$= 0.1 + 0.4 + 0.2$$
$$= 0.7$$

$\therefore \quad P(A' \cup B') = 0.7$

a) Consider the Figure 7.9

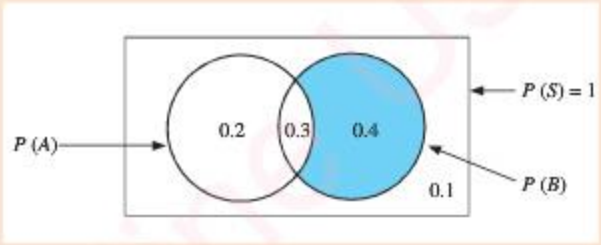

Figure 7.9 Venn diagram 2

The shaded region represents $P(B \cap A')$

$$\therefore P(B \cap A') = 0.4$$

c) Consider the Figure 7.10

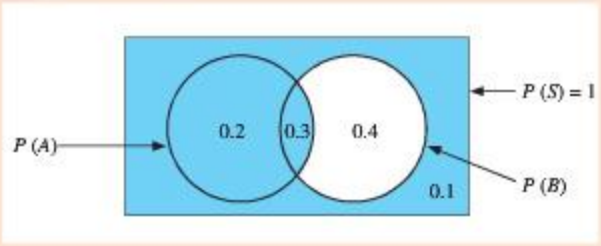

Figure 7.10 Venn diagram 3

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The shaded region represents $P(A \cup B')$

$$P(A \cup B') = 1 - P(B - A)$$
$$= 1 - 0.4$$
$$= 0.6$$
$$P(A \cup B') = 0.6$$

Alternatively

$$P(A \cup B') = \text{Sum of probabilities in the shaded region}$$
$$= 0.1 + 0.2 + 0.3$$
$$= 0.6$$
$$\therefore \quad P(A \cup B') = 0.6$$

Remark: If $P(A)$ represents probability of an event A to occur and $P(A')$ represents probability of an event A not to occur, then $P(A) + P(A') = 1$

Exercise 7.5

1. A bag contains 8 marbles of which 3 are red and 5 are blue. One marble is drawn at random, its colour is noted, and the marble is replaced in the bag. A marble is again drawn from the bag and its colour is noted.
   a) Draw a tree diagram showing the probabilities of each outcome of the two trials.
   b) Find the probability that the marbles drawn will be red followed by blue.
   c) Find the probability that the marbles drawn will be red and blue in any order.
2. A fair coin is tossed and a die is rolled. Find the probability of obtaining:
   a) a head and a number less than 3
   b) a tail and an even number
3. A family has three children. Draw a tree diagram and find the probability that:
   a) all three children are boys
   b) one is a girl and two are boys
   c) there are at least 2 girls
4. Two dice are tossed. Find the probability of getting:
   a) at least one 4
   b) a total score of 6
5. A bag contains 10 marbles in which 7 are red and 3 are yellow. Two marbles are drawn one after another without replacement. Find the probability that:
   a) both marbles are red.
   b) one marble is red and the other is yellow.

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7.4 Mutually exclusive events

Two events are said to be mutually exclusive if they cannot occur together; i.e. they have no elements in common. On the other hand, non-mutual exclusive events can occur together; i.e. they have common elements.

For non-mutual exclusive events, the formula used is:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

where;

$P(A \cup B)$ = Probability of an event A or event B to occur

$P(A \cap B)$ = Probability of both events A and B to occur

For mutual exclusive events, $P(A \cap B) = 0$ since the events cannot occur together and the formula is simplified to:

$$P(A \cup B) = P(A) + P(B)$$

Example 7.16: One card is selected from a pack of 52 playing cards. What is the probability that the card selected will be either a diamond or a black ace.

Solution

$$\text{From } P(\text{diamond}) = \frac{13}{52}$$

$$P(\text{black ace}) = \frac{2}{52}$$

$$P(\text{diamond or black ace}) = P(\text{diamond}) + P(\text{black ace})$$

$$= \frac{13}{52} + \frac{2}{52} = \frac{15}{52}$$

Therefore, the probability that the card selected is either a diamond or a black ace is $\frac{15}{52}$.

Example 7.17: At a certain school there are 30 teachers in which, 10 are Physics teachers, 8 are Chemistry teachers and 5 are Mathematics teachers. If a teacher is selected at random, find probability of getting:

a) a Physics teacher or a Chemistry teacher.

b) a Chemistry teacher or a Mathematics teacher.

Solution

Let A represents an event that Physics teacher is selected, B represents an event that Chemistry teacher is selected and C represents an event that a Mathematics teacher is selected then,

$$P(A) = \frac{n(A)}{n(S)} = \frac{10}{30} = \frac{1}{3},$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{8}{30} = \frac{4}{15},$$

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$$P(C)=\frac{n(C)}{n(S)}=\frac{5}{30}=\frac{1}{6}$$

a) $P(A\cup B)=P(A)+P(B)$

$$=\frac{1}{3}+\frac{4}{15}=\frac{3}{5}$$

Therefore probability of selecting a Physics teacher or a Chemistry teacher is $\frac{3}{5}$

b) $P(A\cup C)=P(B)+P(C)$

$$=\frac{4}{15}+\frac{1}{6}=\frac{13}{30}$$

Therefore probability of selecting a Chemistry teacher or a Mathematics teacher is $\frac{13}{30}$

Example 7.18: A and B are mutually exclusive events such that

$P(A)=\frac{1}{2}$ and $P(B)=\frac{1}{4}$, find

a) $P(A\cup B)$

b) $P(A\cap B)$

c) $P(A'\cap B')$

Solution:

a) $P(A\cup B)=P(A)+P(B)$

$$=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

$\therefore P(A\cup B)=\frac{3}{4}$

b) Since A and B are mutually exclusive events then

$P(A\cap B)=0$

c) $P(A'\cap B')=P(A\cup B)'$

$$=1-P(A\cup B)$$

$$=1-\frac{3}{4}=\frac{1}{4}$$

$\therefore P(A'\cap B')=\frac{1}{4}$

Exercise 7.6

1. There are 5 goldfish, 4 Neon fish, 3 guppies, and 1 glass fish in an aquarium. A fish is selected at random from the aquarium. Find the probability that the selected fish is:
   a) a goldfish or a guppy
   b) a goldfish or a neon fish or a glass fish.
2. Two dice are rolled. Events E_1, E_2, and E_3 are defined as follows:

   E_1 : Getting a sum equal to 10

   E_2 : Getting a double

   E_3 : Getting a sum less than 4

   Find the probabilities of $P(E_1$ or $E_2)$ and $P(E_1$ or $E_3)$.

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3. A card is drawn from a deck of 52 playing cards. What is the probability of drawing a heart or the king of spades?

4. A jar contains three red marbles and four blue marbles. One of the marble is drawn at random. How many possible outcomes are in the sample space? Explain how did you arrive to your answer.

5. A bag contains red, blue, and green counters of the same size and shape. A counter is taken at random from the bag. The probability that it is red is 1.5 times the probability that it is blue, and the probability that it is blue is twice the probability that it is green.

   Find the probabilities that the counter is:
   (a) red (b) blue (c) green.

6. A bag contains 8 discs of which 4 are red, 3 are blue, and 1 is yellow. Calculate the probability that when one disc is drawn from the bag it will be:
   (a) red (b) yellow
   (c) blue (d) not blue

7.5 Independent and dependent events

a) Independent Events

Two events are said to be independent if the occurrence of one of them has no effect on the occurrence of the other. For example, when a fair coin is tossed twice, the outcome of the first toss has no effect on the outcome of the second toss. Therefore, the two events are said to be independent. To find the probability of two independent events which occur together, find the probability of each event occurring separately, and then multiply the probabilities.

When two events A and B are independent, the probability of both to occur is:

$$P(A \text{ and } B) = P(A) \times P(B) \Rightarrow P(A \cap B) = P(A) \times P(B)$$

which is called multiplicative rule for independent events.

Example 7.19: A fair coin is tossed and a single 6 sided die is rolled. Find the probability of obtaining the head side of the coin and rolling a 3 on the die.

Solution

From $P(E) = \dfrac{n(E)}{n(S)}$

$$P(\text{head}) = \frac{1}{2}$$

$$P(\text{head and } 3) = P(\text{head}) \times P(3)$$

$$= \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

Therefore, the $P(\text{head and } 3) = \dfrac{1}{12}$

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Example 7.20: A card is chosen at random from a deck of 52 cards. It is then replaced and a second card is chosen. What is the probability of choosing a jack and then an eight?

Solution

From $P(E) = \frac{n(E)}{n(S)}$

$$P(\text{jack}) = \frac{4}{52}$$

$$P(8) = \frac{4}{52}$$

$$P(\text{jack and } 8) = P(\text{jack}) \times P(8)$$

$$= \frac{4}{52} \times \frac{4}{52} = \frac{16}{2704}$$

Therefore, the $P(\text{jack and } 8) = \frac{1}{169}$

Exercise 7.7

1. A bag contains three white and two red ball. If two balls are picked one at a time with replacement, find the probability that
   a) both balls are red
   b) balls are of different colours
2. Let E_1 and E_2 be the events "heads on the fifth toss" and "head on sixth toss" of a coin, respectively. What is the probability of head on both fifth and sixth tosses (assuming the toss to be fair)
3. If A and B are independent events, prove that
   a) $P(A/B) = P(A)$
   b) $P(B/A) = P(B)$
4. A coin is tossed three times. If A is the event that a head occurs on each of the first two tosses, B is the event that a tail occurs on the third toss and C is the event that exactly two tails occur in the three tosses. Show that A and B are independent events.
5. Two integers are chosen with replacement from the digits 1, 2, 3, 4, 5 and 6. State whether the events described are independent or not and state the reason.

b) Dependent events

Dependent events are those events in which the occurrence of one event affects the occurrence of the other. Two events A and B are said to be dependent if

$$P(A \cap B) = P(A/B) \times P(B)$$

This is called multiplicative rule for dependent events. The term $P(A/B)$ is a notation for conditional probability. It is read as "probability of A given that B has occurred".

For example, if cards are drawn from a pack of 52 playing cards without replacement, the event of getting a diamond in the first draw will cause an impact on the event of getting the diamond in the second draw. That is, the probability of getting a diamond in the first draw will be $\frac{13}{52}$ while the probability of getting a diamond in the second draw will be $\frac{12}{51}$.

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Example 7.21: If A and B are dependent events such that $P(B)=\frac{2}{3}$ and, $P(A\cap B)=\frac{1}{4}$, find $P(A/B)$.

Solution

From $P(A\cap B)=P(A/B)\times P(B)$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}$$

$$P(A/B)=\frac{1}{4}\div\frac{2}{3}$$

$$=\frac{1}{4}\times\frac{3}{2}$$

$$=\frac{3}{8}$$

$$\therefore P(A/B)=\frac{3}{8}$$

Example 7.22: Select two cards from the standard deck of 52 cards without replacement. Find the probability of selecting two kings.

Solution

Let A be the event that the first card selected is king and B be the event that the second card selected is king. Then $\frac{4}{52}$ as there are 4 king cards in a deck.

Now, we do not replace the first card back into the deck. Selecting a second king now, given that the first selected is king; then, $P(B/A)=\frac{3}{51}$ as there are 3 kings left among 51 cards. Hence, the probability of selecting two kings without replacement is:

$$\mathrm{P}(A \text{ and } B)=P(A)\times P(B/A)$$

$$=\frac{1}{13}\times\frac{1}{17}$$

$$\therefore P(A\cap B)=\frac{1}{221}$$

Example 7.23: A committee consists of four women and three men. The committee will randomly select two people to attend a conference in Nairobi. The first person to be selected will not be considered in the second selection. Find the probability that both are women.

Solution

Let A be the event that the first person selected is a woman and B be the event that the second person selected is a woman. Then, $P(A)=\frac{4}{7}$ and $P(B/A)=\frac{3}{6}$; i.e. there are 4 women in the committee of 7 people. Now, as we select a woman as the first person to attend the conference, we cannot select her as a second person to attend the conference. Therefore, there are 6 people left to be selected from the committee and only 3 of them are women.

So to find the probability of selecting both women is:

$$\mathrm{P}(A \text{ and } B)=P(A)\times P(B/A)$$

$$=\frac{4}{7}\times\frac{3}{6}$$

$$=\frac{4}{7}\times\frac{3}{6}$$

$$\therefore P(\cap B)=\frac{2}{7}$$

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Exercise 7.8

1. Given that, for the dependent events A and B:
   $n(A)=32, n(B)=28$ and $n(A\cap B)=15$
   Calculate:
   a) $P(A/B)$
   b) $P(B/A)$
2. Select two cards from the standard deck of 52 cards without replacement. Find the probability of selecting two diamonds.
3. A bag contains 3 red balls, 5 black balls and 2 green balls. Two balls are drawn one at a time and without replacement. What is the probability of drawing the first black and the second red?
4. Given that $P(A/B)=\frac{3}{8}$ and $P(B)=\frac{1}{3}$, where A and B are dependent events, then find
   a) $P(A\cap B)$
   b) $P(A/B)'$
5. At a certain secondary school, the probability that a student takes Biology and Geography is 0.76. The probability that a student takes Biology is 0.84. What is the probability that a student takes Geography given that he is taking Biology?

7.6 Applications of probability

In areas where statistical analysis is based, probability is used to predict uncertainty, make estimates and predict results of a research investigation.

Prediction and estimation helps in decision making and hence planning and controlling for the future.

Below are some examples of the kinds of problems that can be handled using probability.

a) Cyclists trying to minimize the chance of accident. They decide to check if some of the parts in their bicycles are in good order or not, and other factors which may be significant causes of accident.

b) Campaign managers for certain political parties use results from a recent poll taken in the country to know the chance that their candidate would win if the election was held on that specific time.

What decisions are you taking after investigating all? Why?

They affect decisions about our health, citizenship, parenthood, jobs, financial concerns, and many other important matters. Today, an informed person must have some facility for dealing with data and making intelligent decisions based on quantitative arguments that involve uncertainty or chance.

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Example 7.24: The success of a small poultry project depends on the availability of good rationed feed and heat for the chicks. It was found that the probability of the availability of good rationed feed and heat, and the starting capital is 0.52. If the probability of success of good rationed feed and heat for the project given that a starting capital is available is 0.85, find the probability of the availability of the starting capital?

Solution

From the example given, it is observed that the availability of good rationed feed and heat depends on availability of capital. Thus, this forms dependent events.

Let A be the event that good rationed feed and heat are available and B be the event that the starting capital is available. Then, the probability that good rationed feed and heat, and starting capital are available is $P(A \cap B) = 0.52$. The probability that good rationed feed and heat given that a starting capital is available is $P(A/B) = 0.85$

The probability of the availability of starting capital $P(B) = ?$

Since probability $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

or $P(B) = \dfrac{P(A \cap B)}{P(A/B)}$, it implies that

$$P(B) = \frac{0.52}{0.85} = 0.61$$

Exercise 7.9

1. If the probability of meeting a building contract date for a firm is 0.65, the probability of a good weather is 0.45 and probability of meeting the date and good weather is 0.351. Calculate the probability that,
   a) there was a good weather given that the contract date was met
   b) the contract date was not met given that there was a good weather.
2. A company has taken a ballot of its employees, asking them to choose between two alternative new schemes A and B for claiming travelling expenses.70% of the employees supports scheme A, 60% of these owns a car. If only 25% of those who supports scheme B own a car, calculate the probability that if an employee is chosen at random, he will be;
   a) a scheme A supporter given that he owns a car.
   b) a scheme B supporter given that he does not own a car.
3. A factory has a machine shop in which two machines $(M_1 \text{ and } M_2)$ produce 250 cm aluminium tubes. An inspector is equally likely to sample tubes from $(M_1 \text{ and } M_2)$. The probability for the defective rates from the two machines $(M_1 \text{ and } M_2)$ are 8% and 12%, respectively. What is the probability that a tube selected

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by the inspector comes from M_2 given that it is defective?

4. The success of a larger poultry depends on the availability of good rationed feed and heat for the chicks. It was found that the probability of the availability of a starting capital is 82%. Also the probability of the availability of good feed and heat, and starting capital is 64%. Find the probability of success of good rationed feed and heat for the project given that a starting capital is available.

5. A company tenders for two contracts A and B. The probability that it will obtain contract A is 0.83. The probability that it will obtain both contracts A and B is 0.64. Find the probability that the company will obtain contract B given that it will obtain contract A.

Revision exercise 7

1. How many 4 digit numbers greater than 5000 can be found using digits 2, 5, 6, 8 and 9 without repetition.
2. All letters in the word UNIVERSAL are to be arranged without repetition such that 3 letters in the middle are vowels. How many ways can this be done?
3. How many 3 digit numbers can be formed using the digits 2, 5, 6, 8 and 9?
4. Evaluate each of the following

   a) ${}^{16}C_{14}$ b) ${}^{22}C_{12}$ c) ${}^{5}C_{0}$

   d) ${}^{11}P_{11}$ e) ${}^{15}P_{4}$

5. Find the value of n given that:

   a) $3\left({}^{n}C_{6}\right) = {}^{n}C_{5}$ b) $3\left({}^{n}C_{3}\right) = {}^{n}C_{2}$

6. Form 3C is looking for a new monitor. The probability of Asha becoming a monitress is 50% and the probability of Fatuma becoming a monitress is 40%. What is the probability of either Asha or Fatuma becoming the monitress?
7. A bag contains 2 yellow balls, 3 green balls, 5 red balls, and 6 black balls. Two balls are drawn one at a time and without replacement. What is the probability of drawing the first red and the second black?

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8. Use the following information to determine whether the events A and B are mutually exclusive or not: $P(A)=\frac{1}{4}$, $P(B)=\frac{1}{3}$, $P(A \text{ or } B)=\frac{7}{12}$.

9. Calculate the probability of the indicated event, given the following information: Shaban and Neema are the two competitors in the final of the long jump competition. The probability that Shaban wins the competition is $\frac{2}{7}$, and the probability that Neema wins is $\frac{1}{5}$, calculate the probability that Shaban or Neema wins the competition.

10. In a group of students, the probability that a student is chosen at random walking to school is 0.35 and the probability that a student has blonde hair is 0.2. If the probability that a student's walks to school or has blonde hair is 0.45, calculate the probability that a student has blonde hair and walks to school.

11. At Msamvu Secondary School, the probability that a student takes Mathematics and Chemistry is 0.087. The probability that a student takes Mathematics is 0.68. What is the probability that a student takes Chemistry given that the student is taking Mathematics?

12. The probability that Ally and Mary will be alive in twenty years to come is 0.65 and 0.70, respectively. Find:
    a) the probability that both will be alive in twenty years to come.
    b) the probability that either Ally or Mary will be alive in twenty years to come.

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Chapter Eight

Trigonometry

Introduction

Trigonometry is the study of the relationship between the sides and angles of a triangle. The knowledge of trigonometry can be applied in various real-life situations such as in solving problems related to astronomy, navigation, and in buildings construction. Generally, trigonometry has great practical importance to builders, architects, surveyors, engineers, and in many other fields. In this chapter, you will learn how to derive and apply trigonometrical ratios and identities, derive and use sine and cosine rules, evaluate trigonometric ratios by using computer packages, draw graphs and perform calculus (differentiation and integration) of trigonometrical functions.

8.1 Trigonometric ratios

Consider ΔORS in Figure 8.1 from which the sine, cosine, and tangent ratios can be deduced.

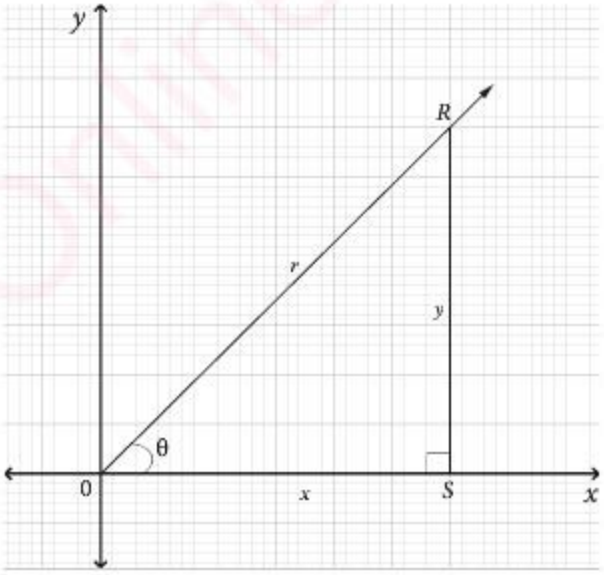

Figure 8.1 A right-angled triangle ORS.

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From Figure 8.1:

$\overline{SR}$ is called the opposite side to angle θ, whose length is denoted by y units.

$\overline{OS}$ is called the adjacent side to angle θ, whose length is denoted by x units.

$\overline{OR}$ is called the hypotenuse whose length is denoted by r units.

$$\sin\theta = \frac{\text{Length of opposite side}}{\text{Length of hypotenuse side}} = \frac{y}{r}$$

$$\cos\theta = \frac{\text{Length of adjacent side}}{\text{Length of hypotenuse side}} = \frac{x}{r}$$

$$\tan\theta = \frac{\text{Length of opposite side}}{\text{Length of adjacent side}} = \frac{y}{x}$$

The terms $\sin\theta$, $\cos\theta$, and $\tan\theta$ are abbreviations of the sine of angle θ, cosine of angle θ, and tangent of angle θ, respectively.

Reciprocals of sine,c osine,an d tangent

$$a)\ \frac{1}{\sin\theta} = \text{cosec}\theta$$

$$b)\ \frac{1}{\cos\theta} = \sec\theta$$

$$c)\ \frac{1}{\tan\theta} = \cot\theta$$

The terms $\text{cosec}\theta$, $\sec\theta$, and $\cot\theta$ are abbreviations of cosecant of angle θ, secant of angle θ, and cotangent of angle θ, respectively.

The relationship between
$\sin\theta$, $\cos\theta$ and $\tan\theta$

Using ΔORS in Figure 8.1, $\sin\theta = \frac{y}{r}$,

$\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$.

Then,

$$\frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x},\ \textit{but}\ \frac{y}{x} = \tan\theta$$

Therefore, $\tan\theta$ can be defined in terms of $\sin\theta$ and $\cos\theta$ as $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

This is true for any angle θ having any magnitude. Similarly $\frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{y}$

But $\frac{x}{y} = \frac{1}{\tan\theta} = \cot\theta$.

Therefore,

$$\cot\theta = \frac{\cos\theta}{\sin\theta}.$$

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Rotated triangle inside the circle

Consider ΔOPQ in Figure 8.2.

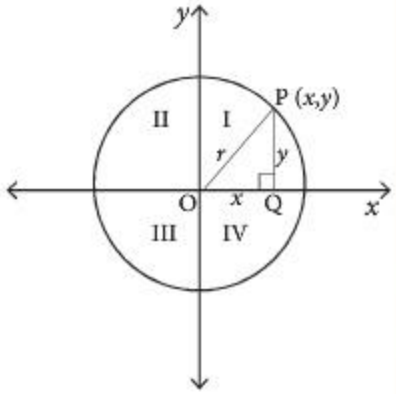

Figure 8.2 Rotated triangle inside the circle

The trigonometric ratios can be positive or negative depending on the quadrant in which $\overline{OP}$ lies. Also, the values of the ratios are related to those of the corresponding acute angles.

Quadrant I

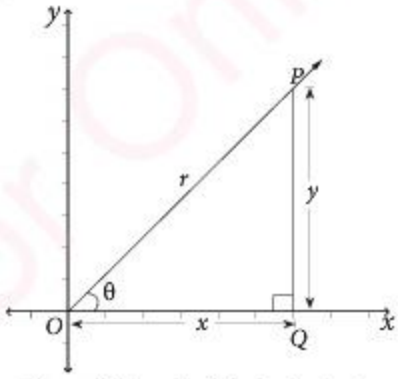

Figure 8.3 An angle in the first quadrant

Using the sides of ΔOPQ in Figure 8.3, we have;

$$\sin\theta = \frac{y}{r}\,(\text{Positive})$$

$$\cos\theta = \frac{x}{r}\,(\text{Positive})$$

$$\tan\theta = \frac{y}{x}\,(\text{Positive})$$

All the trigonometric ratios in the first quadrant are positive as per corresponding axes of x and y.

Quadrant II

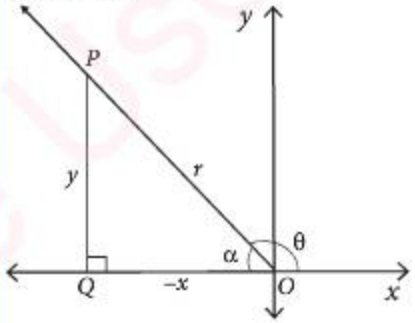

Figure 8.4 An angle in the second quadrant

Using the sides of ΔOPQ in Figure 8.4, we have;

$$\sin\theta = \frac{y}{r} = \sin\alpha \qquad \text{(Positive)},$$

$$\cos\theta = \frac{-x}{r} = -\cos\alpha \qquad \text{(Negative)},$$

$$\tan\theta = \frac{y}{-x} = -\tan\alpha \qquad \text{(Negative)},$$

where $\alpha = 180° - \theta$.

In the second quadrant, the sine is positive while the cosine and the tangent are negative as per corresponding axes of x and y.

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Example 8.1: Express the following in terms of the trigonometric ratios of the acute angles

a) $\sin 150°$

b) $\cos 130°$

c) $\tan 120°$

Solution

a) $\sin 150° = \sin(180° - 150°) = \sin 30°$

b) $\cos 130° = -\cos(180° - 130°) = -\cos 50°$

c) $\tan 120° = -\tan(180° - 120°) = -\tan 60°$

Quadrant III

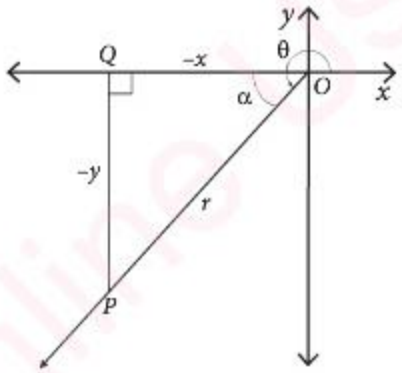

Figure 8.5 An angle in the third quadrant

Using the sides of triangle OPQ in Figure 8.5, the sine, cosine and tangent ratios become;

$$\sin\theta = \frac{-y}{r} = -\sin\alpha \quad \text{(Negative)},$$

$$\cos\theta = \frac{-x}{r} = -\cos\alpha \quad \text{(Negative)},$$

$$\tan\theta = \frac{-y}{-x} = \tan\alpha \quad \text{(Positive)},$$

where $\alpha = \theta - 180°$.

In the third quadrant, the sine and cosine are negative while the tangent is positive since both axes are negative.

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Example 8.2: Express the following in terms of the trigonometric ratios of the acute angles

a) $\sin 210°$

b) $\cos 240°$

c) $\tan 250°$

Solution

a) $\sin 210° = -\sin(210° - 180°) = -\sin 30°$

b) $\cos 240° = -\cos(240° - 180°) = -\cos 60°$

c) $\tan 250° = \tan(250° - 180°) = \tan 70°$

Quadrant IV

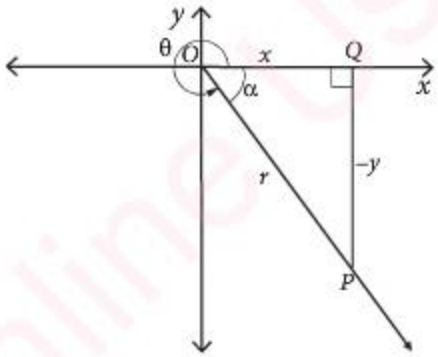

Figure. 8.6 An angle in the fourth quadrant

From ΔOPQ in Figure 8.6, the sine, cosine and tangent ratios become;

$$\sin\theta = \frac{-y}{r} = -\sin\alpha \qquad \text{(Negative)},$$

$$\cos\theta = \frac{x}{r} = \cos\alpha \qquad \text{(Positive)},$$

$$\tan\theta = \frac{-y}{x} = -\tan\alpha \qquad \text{(Negative)},$$

where $\alpha = 360° - \theta$.

In fourth quadrant, the sine and tangent are negative while the cosine is positive since x – axis is positive and y – axis is negative.

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Example 8.3: Express each of the following in terms of the trigonometric ratios of acute angles:

a) $\sin 330°$

b) $\cos 330°$

c) $\tan 330°$

Solution

a) $\sin 330° = -\sin(360° - 330°) = -\sin 30°$

b) $\cos 330° = \cos(360° - 330°) = \cos 30°$

c) $\tan 330° = -\tan(360° - 330°) = -\tan 30°$

Exercise 8.1

1. Write down the sign of each of the following trigonometric ratios and give reasons.
   a) $\sin 248°$
   b) $\cos 320°$
   c) $\tan 140°$
2. Express each of the following in terms of the trigonometrical ratios of the acute angles:
   a) $\sin 162°$
   b) $\tan 251°$
   c) $\cos 346°$
3. Express each of the following in terms of the ratios of the acute angles:
   a) $\text{cosec} 215°$
   b) $\cot 130°$
   c) $\sec 335°$
4. State the quadrant of each of the following angles:
   a) $192°$
   b) $315°$
   c) $-80°$
5. Prove that $\sin^2\theta + \cos^2\theta = 1$, where θ is an acute angle

Activity 8.1

Use a non-programmable scientific calculator to compute

a) $\sin 20°$ and $\cos 70°$
b) $\cos 30°$ and $\sin 60°$
c) explain what you have noticed in each case (a) and (b).

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Complementary angles

Two angles α and β (see Figure 8.7) are said to be complementary if their sum is $90°$.

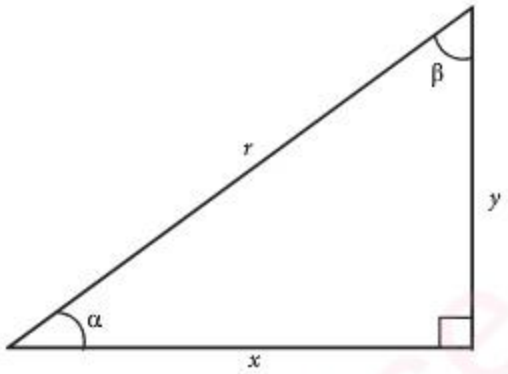

Figure 8.7 A right-angled triangle showing complementary angles

With reference to the triangle in Figure 8.7, we have;

$\alpha + \beta + 90° = 180°$ (sum of interior angles of the triangle).

It implies that

$\alpha + \beta = 90°$ where, α and β are complementary angles.

Thus, $\alpha = 90° - \beta$ and $\beta = 90° - \alpha$

$\sin\alpha = \frac{y}{r}$, $\sin\beta = \frac{x}{r}$, and

$\cos\alpha = \frac{x}{r}$, $\cos\beta = \frac{y}{r}$. .

Therefore, $\sin\alpha = \cos\beta$ but $\beta = 90° - \alpha$, and hence, $\sin\alpha = \cos(90° - \alpha)$

$\cos\alpha = \sin\beta$ but $\beta = 90° - \alpha$, and hence $\cos\alpha = \sin(90° - \alpha)$

Remark: The sine of an acute angle is equal to the cosine of its complement and vice versa

Example 8.4: Express each of the following in terms of the corresponding complementary angles.

a) $\sin 60°$

b) $\cos 35°$

Solution

a) $\sin 60° = \cos(90° - 60°) = \cos 30°$

b) $\cos 35° = \sin(90° - 35°) = \sin 55°$

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Special angles

The trigonometric ratios for angles such as $0°, 30°, 45°, 60°, 90°, 180°, 270°$, and $360°$ are fequently used. Besides, their trigonometric ratios can be obtained without necessarily using calculating devices. These kind of angles are known as special angles and their trigonometric ratios are worthy memorizing.

Trigonometric ratios for $30°$ and $60°$

Consider an equilateral triangle ABC of side 2 units.

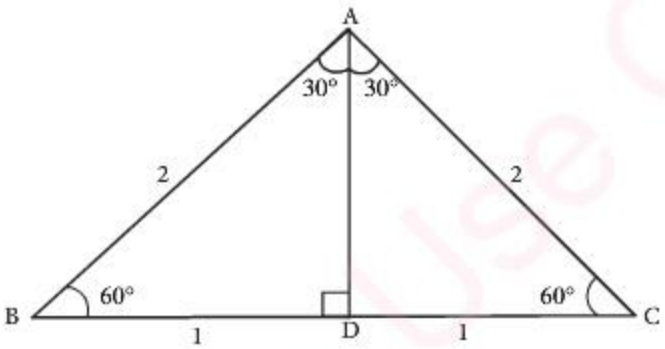

Figure 8.8 Equilateral triangle

In ΔABC of Figure 8.8, the altitude AD bisects line segment BC, using the Pythagoras theorem, we get;

$$\overline{AB}^2 = \overline{BD}^2 + \overline{AD}^2$$

$$2^2 = 1^2 + \overline{AD}^2$$

$$\overline{AD}^2 = 4 - 1 = 3$$

$$\overline{AD} = \sqrt{3}$$

Hence, $$\sin 60° = \cos 30° = \frac{\overline{AD}}{\overline{AB}} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \sin 30° = \frac{\overline{BD}}{\overline{AB}} = \frac{1}{2}$$

$$\tan 60° = \frac{\overline{AD}}{\overline{AB}} = \frac{\sqrt{3}}{1} = \sqrt{3} \text{ and } \tan 30° = \frac{\overline{BD}}{\overline{AD}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

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Trigonometric ratios for angle $45°$

Consider an isosceles triangle ABC

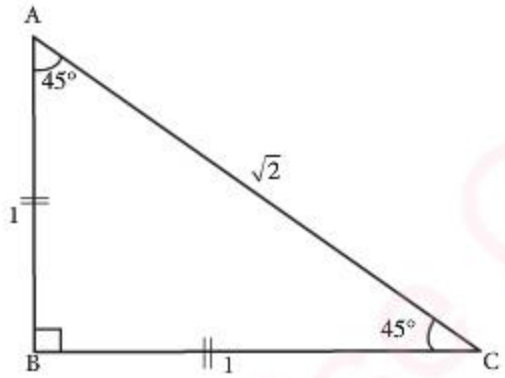

Figure 8.9 An isosceles triangle ABC

Let $\overline{AB} = \overline{BC} = 1$ unit. Now, by Pythagoras theorem, we have;

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$$

$$\overline{AC}^2 = 1^2 + 1^2$$

$$\overline{AC}^2 = 2$$

$$\therefore \overline{AC} = \sqrt{2}$$

Then, $\sin 45° = \dfrac{\overline{AB}}{\overline{AC}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$$\cos 45° = \frac{\overline{BC}}{\overline{AC}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{\overline{AB}}{\overline{BC}} = \frac{1}{1} = 1$$

Hence, the sine, cosine and tangent of special angles are summarized in Table 8.1.

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Table 8.1 Summary for Values of Special Angles

Angles	0°	30°	45°	60°	90°	180°	270°	360°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0	1
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	0	∞	0

Example 8.5: Evaluate $2\cos 60°\sin 45° + 3\tan 60°\cos 30°$ leaving the answer in the surd form.

Solution

$$2\cos 60°\sin 45° + 3\tan 60°\cos 30° = 2\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + 3\left(\sqrt{3}\right)\frac{\sqrt{3}}{2}$$

$$= 2\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{9}{2}$$

$$= \frac{1}{\sqrt{2}} + \frac{9}{2} = \frac{\sqrt{2}}{2} + \frac{9}{2}$$

$$= \frac{\sqrt{2}+9}{2}$$

$$\therefore \quad 2\cos 60°\sin 45° + 3\tan 60°\cos 30° = \frac{\sqrt{2}+9}{2}$$

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Exercise 8.2

1. Find the values of each of the following leaving answers in surds.
   a) $\tan 240°$
   b) $\cos(-135)°$
   c) $\sin(-120°)$

2. If A and B are two acute angles of the right angled triangle, show that $\sin A = \cos B$.

3. Find the values of each of the following, leaving surds in your answers.
   a) $\cot(-690°)$
   b) $\operatorname{cosec} 240°$
   c) $\sec 405°$

4. Given that α and β are two complementary angles and $\cos\alpha = \frac{\sqrt{3}}{2}$, find the value of $\sin\beta$ without using calculating devices.

5. Find the values of each of the following without using calculating devices.
   a) $\frac{2\tan 45°}{\sec 60°} + \frac{3\operatorname{cosec} 30°}{\cot 45°}$
   b) $\frac{1+\tan 30°}{1+\tan 60°}$

Evaluating trigonometric ratios using a computer spreadsheet package

Trigonometric ratios may also be evaluated using appropriate computer package. In this book, the Excel spreadsheet package is used to demonstrate on how to evaluate different trigonometric ratios using computer spreadsheet package.

In order to evaluate trigonometric ratios of any angle, use the following steps.

Step 1: Open "Microsoft office Excel.

Step 2: Click "Formulae" from the top bar.

Step 3: Click "Math & Trig" functions from the top bar of spreadsheet.

Step 4: Select "SIN" or "COS" or "TAN" from the options displayed in "Math & Trig" function.

Step 5: In the pop-up window titled "Function arguments", type the angle to be evaluated in the space preceded by the word "Number".

Step 6: Press "OK" to obtain the result.

Remark: When using the spreadsheet to evaluate trigonometric functions, the answer obtained will depend on whether the argument (angle) is in degrees or radians. If the argument (x) is in degrees, multiply it by $\frac{\pi}{180°}$ that is $\frac{\pi x}{180°}$ or use the "RADIANS" function to convert it to radians.

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Example 8.6: Use spreadsheet package to evaluate $\sin 45°$.

Solution

a) Open "Microsoft office Excel"

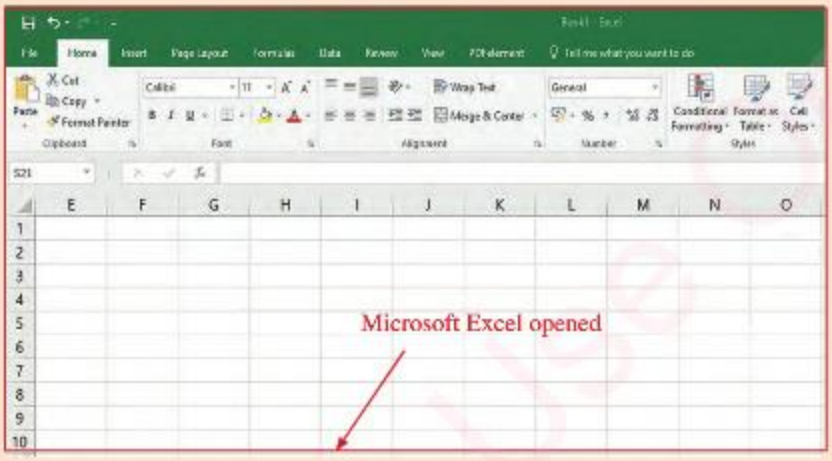

Figure 8.10 Microsoft office Excel

b) Click "Formulas" from the top bar of the spreadsheet.

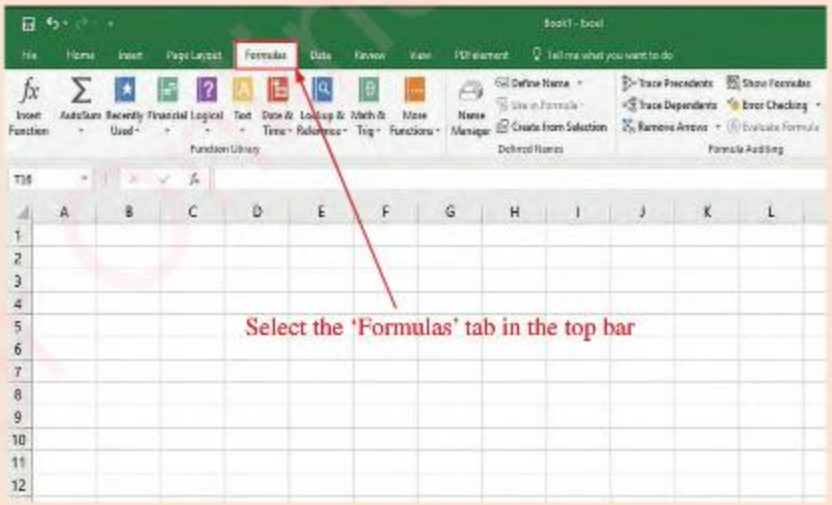

Figure 8.11 Microsoft Excel window showing how to create formulas

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c) Click "Math & Trig" function (see Figure 8.12) to see the options from the trigonometric function and then select the "SIN" from the given options.

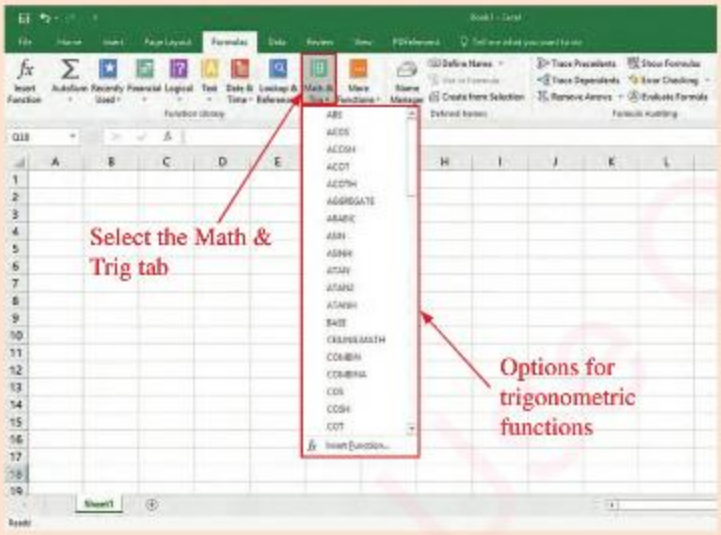

Figure 8.12 Microsoft Excel window showing how to select the math and trig function

d) In the pop-up window titled "Function argument" which will appear, type "45" in the space provided that is preceded by the number. See Figure 8.13 and Figure 8.14.

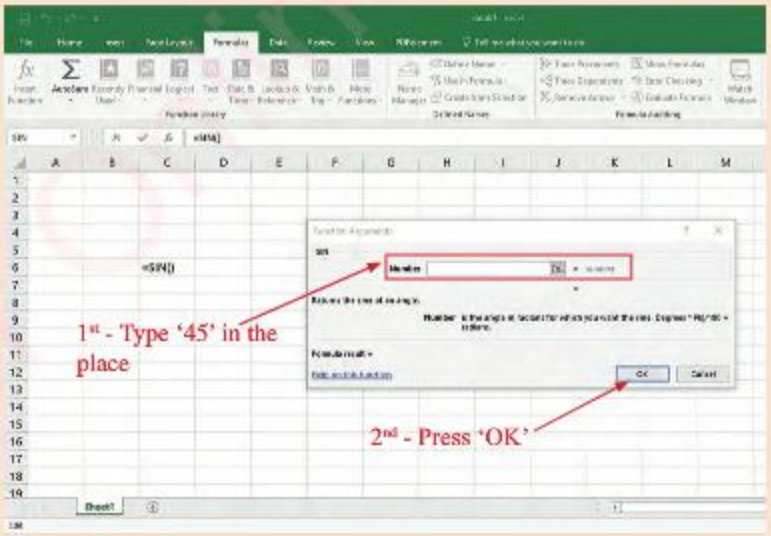

Figure 8.13 Microsoft Excel window showing function arguments

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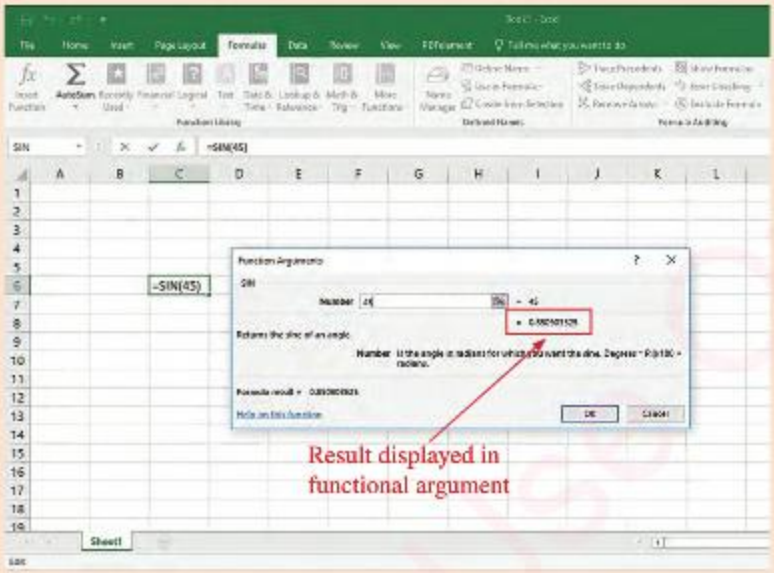

Figure 8.14 Microsoft Excel window displaying result in the functional argument

e) After pressing "OK", the result will be displayed in one of the cells in the spreadsheet, which is 0.850903525 as seen in Figure 8.15.

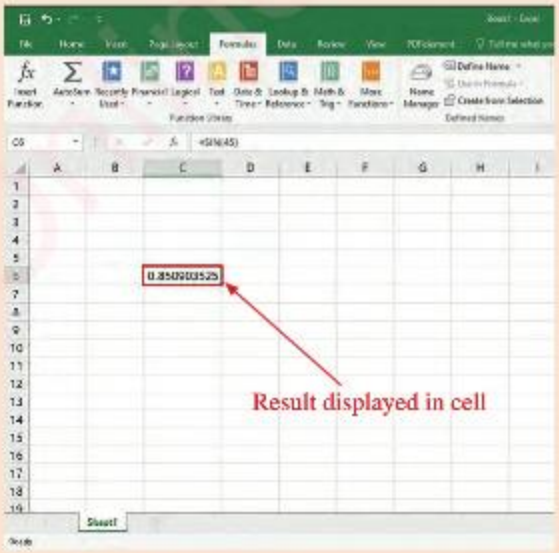

Figure 8.15 Microsoft Excel window showing result displayed in a cell

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Alternative procedure of using spreadsheet may also be applied. Consider example 6; evaluate $\sin 45$.

Steps:

a) Open "Microsoft office excel".

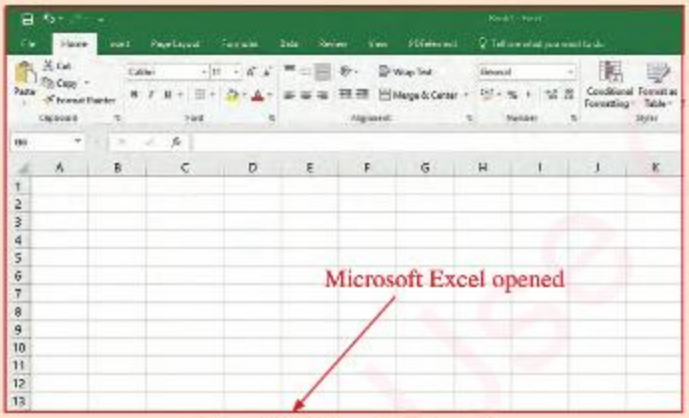

Figure 8.16 Microsoft office excel

b) Choose any cell in the spreadsheet and then type "=" sign.

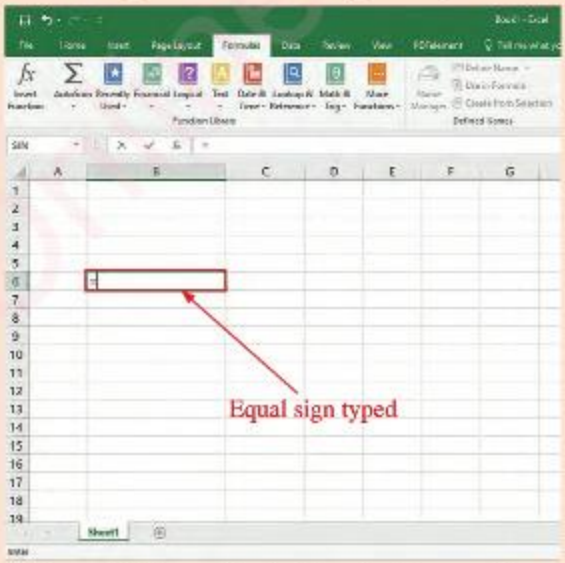

Figure 8.17 Microsoft Excel window displaying equal sign

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c) Type the word "SIN" followed by open bracket "(" in the chosen cell.

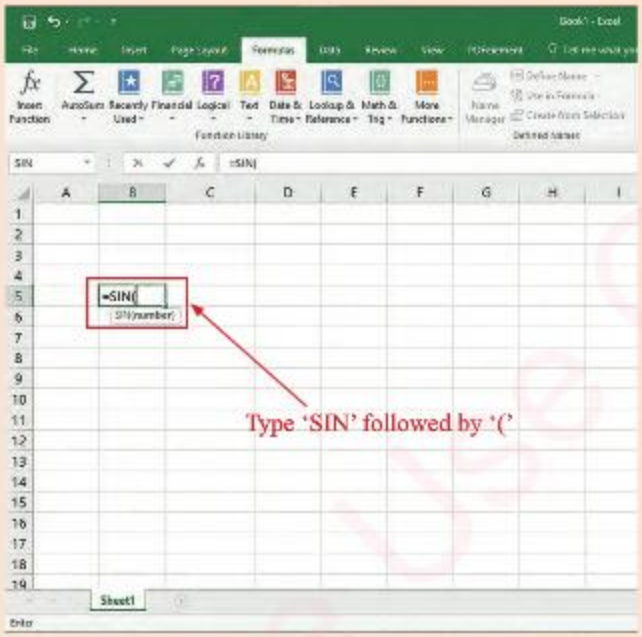

Figure 8.18 Microsoft Excel window showing Function arguments

d) Type the given angle i.e. 45 followed by the close bracket ")"

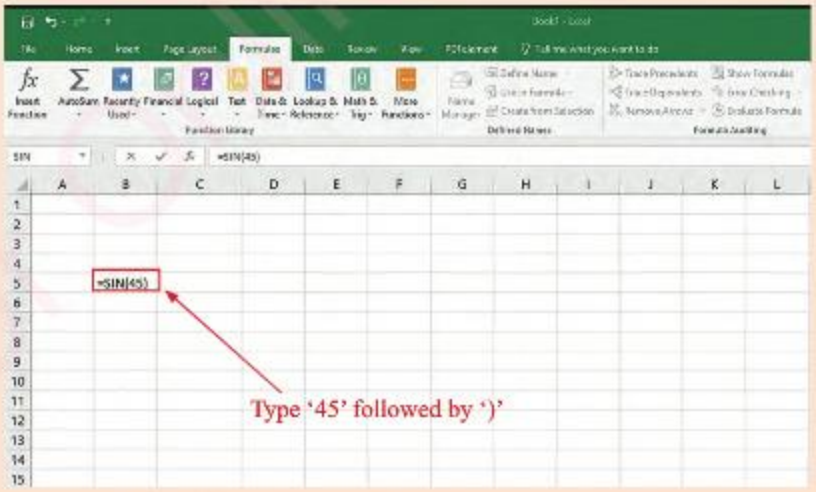

Figure 8.19 Microsoft Excel window showingFunction arguments

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e) Press "ENTER" to obtain the result.

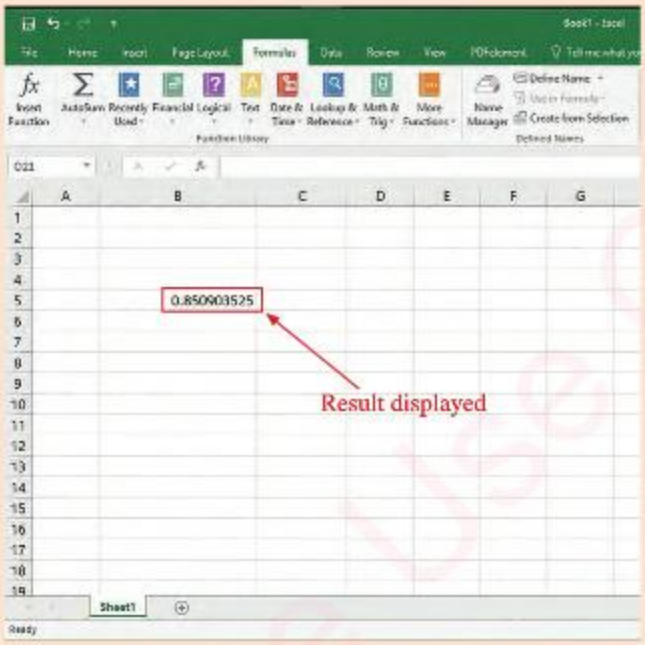

Figure 8.20 Microsoft Excel window showing a display of the result

Therefore, the result obtained is 0.850903525 which is in radians.

Activity 8.2

1. Evaluate each of the following using a spreadsheet package:
   a) $\cos\left(\frac{\pi}{6}\right)$ b) $\tan\left(\frac{\pi}{3}\right)$
2. Evaluate each of the following using a spreadsheet package:
   a) $\cos 30°$ b) $\sin 45°$
3. Evaluate each of the following using a spreadsheet package
   a) $\sin\left(\frac{\pi}{6}\right)$ b) $\tan\left(\frac{5\pi}{4}\right)$
4. Using a spreadsheet package, evaluate each of the following:
   a) $\cos 330°$ b) $\sin 560°$
5. Evaluate each of the following using a spreadsheet package:
   a) $\tan\left(\frac{12\pi}{5}\right)$ b) $\cos 143.5°$

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8.2 The sine and cosine rules

a) The Sine rule

Consider triangle ABC which is not a right angled triangle, in which A,B and C are the angles while the small letters a, b and c denote the sides of the triangle.

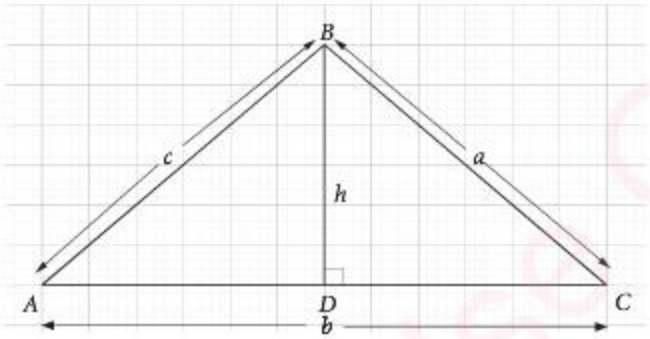

Figure 8.21 Illustration for derivation of the sine rule using the sides $\overline{AB}$ and $\overline{BC}$

By drawing a line perpendicular to side AC from angle B, triangles ABD and CBD are formed. The two triangles share the same height h.

From ΔABD, we have;

$$\sin A = \frac{h}{c} \Rightarrow h = c\sin A \qquad (8.1)$$

and from ΔCBD, we have;

$$\sin C = \frac{h}{a} \Rightarrow h = a\sin C \qquad (8.2)$$

Equating equations (8.1) and (8.2) for h gives;

$$c\sin A = a\sin C \qquad (8.3)$$

Dividing by $\sin A \sin C$ on both sides of equation (8.3) we obtain;

$$\frac{c\sin A}{\sin A\sin C} = \frac{a\sin C}{\sin A\sin C}$$

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$$\frac{c}{\sin C}=\frac{a}{\sin A} \tag{8.4}$$

Suppose ΔABC is divided by drawing a perpendicular line from angle A to the side BC

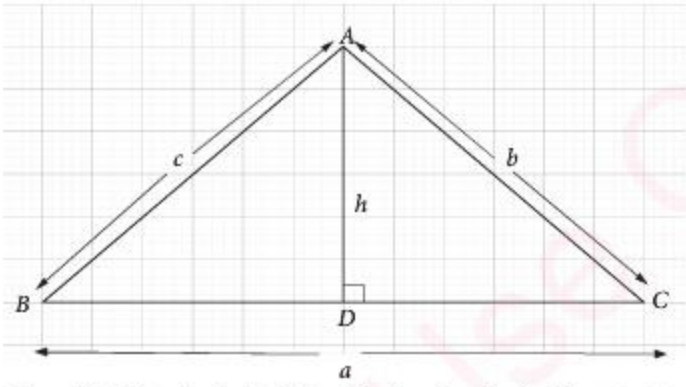

Figure 8.22 Illustration for derivation of the sine rule using the sides $\overline{AB}$ and $\overline{AC}$

Then, the height h can also be obtained by dealing with triangles BAD and CAD

Using ΔBAD, we have;

$$\sin B=\frac{h}{c}\Rightarrow h=c\sin B \tag{8.5}$$

and from ΔCAD, we have;

$$\sin C=\frac{h}{b}\Rightarrow h=b\sin C \tag{8.6}$$

Equating equation (8.5) and (8.6) gives

$$c\sin B=b\sin C \tag{8.7}$$

Dividing by $\sin B\sin C$ both sides of equation (8.7) we get

$$\frac{c\sin B}{\sin B\sin C}=\frac{b\sin c}{\sin B\sin C}$$

$$\frac{c}{\sin C}=\frac{b}{\sin B} \tag{8.8}$$

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From equations (8.4) and (8.8)

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \qquad (8.9)$$

Equation (8.9) is called **Sine rule.**

Remark: Sine rule is used only when, either:

a) two angles and one side are given, or
b) two sides and one angle are given.

Example 8.7: In the triangle RST, side $\overline{ST}=8$ cm, $S\hat{R}T=51°$ and $R\hat{S}T=68$. Find the length of $\overline{RT}$.

Solution: Given $\overline{ST}=8$ cm, $S\hat{R}T=51°$, and $R\hat{S}T=68°$.

To find the length of side $\overline{RT}$, consider the Figure 8.23.

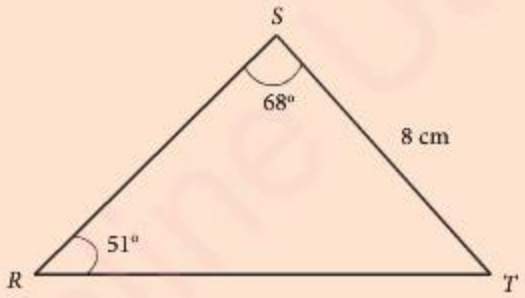

Figure 8.23 The triangle RST

Using the sine rule $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{sin C}$

Then $\frac{8}{\sin 51°}=\frac{\overline{RT}}{\sin 68°}$

$$\overline{RT}=\frac{\mathbf{8\sin 68°}}{\mathbf{\sin 51°}}$$

$$\overline{RT}=9.545$$

Therefore, the length of the side RT is 9.545 cm

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Example 8.8: In triangle ABC, $a = 2.5$ cm, $b = 4.5$ cm and $A\hat{B}C = 48°$. Find the value of an angle $B\hat{A}C$.

Solution

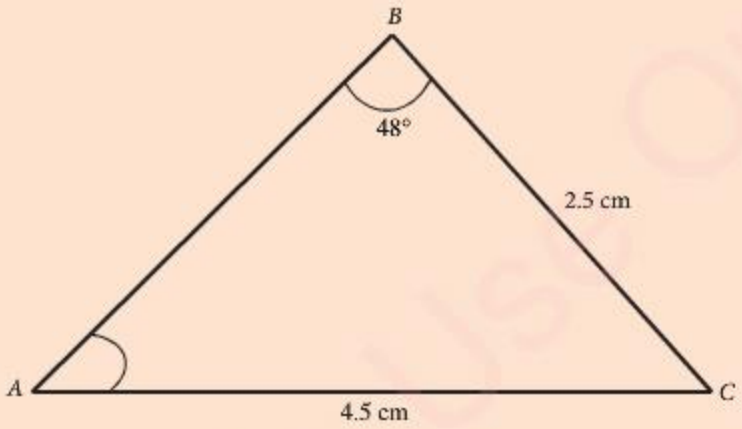

Figure 8.24 The triangle ABC

Applying the sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{2.5 \text{ cm}}{\sin A} = \frac{4.5 \text{ cm}}{\sin 48°}$$

$$\sin A = \frac{2.5 \times \sin 48°}{4.5}$$

$$\sin A = 0.41286$$

$$A = \sin^{-1}(0.41286) = 24.4°$$

$\therefore \quad B\hat{A}C = 24.4°$

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Exercise 8.3

1. Calculate to two decimal places the size of the angles $R\hat{P}Q$ and $P\hat{R}Q$ in the triangle PQR where $\overline{PR} = 4$ cm, $\overline{QR} = 2.35$ cm, and $P\hat{Q}R = 48°$.
2. In a triangle DEF, $\overline{DE} = 5.5$ cm and the size of angles $E\hat{D}F$ and $D\hat{E}F$ are $53°$ and $68°$, respectively. Calculate the length of side DF.
3. Calculate to two decimal places, the size of the angles $C\hat{A}B$ and $A\hat{C}B$ in a triangle ABC, where $\overline{AC} = 6$ cm, $\overline{BC} = 8$ cm and $A\hat{B}C = 40°$.
4. In the triangle ABC, $\overline{AB} = 9$ cm, $B\hat{A}C = 51°$ and $A\hat{C}B = 39°$, find $\overline{BC}$.
5. In the triangles PQR, $P\hat{R}Q = 42.5°$, $P\hat{Q}R = 82.4°$ and 16 cm. Find $\overline{PQ}$.
6. In the triangle ABC, $\overline{AB} = 20$ cm, $\overline{BC} = 18.2$ cm, and $\overline{AC} = 32$ cm. Can the sine rule be used to find $B\hat{A}C$? If the answer is Yes, write down the two ratios of the sine rule that can be used. If the answer is No, give the reason.

b) *The Cosine rule*

Consider triangle ABC, in which $\overline{BD}$ is a perpendicular line drawn from B to the side $\overline{AC}$. Let $\overline{BD} = h$, $\overline{AD} = x$ and $\overline{DC} = b - x$.

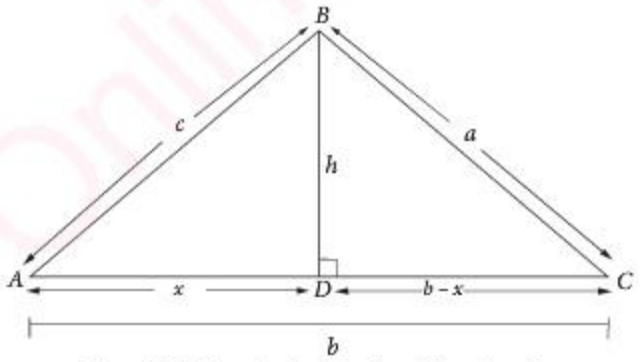

Figure 8.25 Illustration for derivation of the cosine rule

Now, the right-angled triangles ABD and CBD are used to derive the cosine rule.

From ΔABD, using the Pythagoras theorem gives;

$$h^2 = c^2 - x^2 \qquad (8.10)$$

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From ΔCBD, using the Pythagoras theorem gives

$$h^2 = a^2 - (b-x)^2 \qquad (8.11)$$

Equating equations (8.10) and (8.11) for h^2 gives:

$$\begin{aligned} c^2 - x^2 &= a^2 - (b-x)^2 \\ &= a^2 - (b^2 - 2bx + x^2) \\ &= a^2 - b^2 + 2bx - x^2 \end{aligned}$$

So $c^2 = a^2 - b^2 + 2bx \qquad (8.12)$

But from ΔABD, $x = c\cos A$

So, equation (8.12) becomes

$$c^2 = a^2 - b^2 + 2bc\cos A \qquad (8.13)$$

Rearranging equation (8.13) gives:

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad (8.14)$$

Also, when the altitude is drawn from A or from C, the similar expressions for the other sides of the triangle ABC are obtained as;

$$b^2 = a^2 + c^2 - 2ac\cos B \qquad (8.15)$$

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad (8.16)$$

Hence, the equations (8.14), (8.15) and (8.16) are called **Cosine rule.**

Example 8.9: In ΔABC, $\overline{BC} = 0.1$ m, $\overline{AC} = 0.12$ m,, and $\hat{C} = 72°$, find $\overline{AB}$

Solution: Given $\overline{BC} = 0.1$ m, $\overline{AC} = 0.12$ m, and $\hat{C} = 72°$

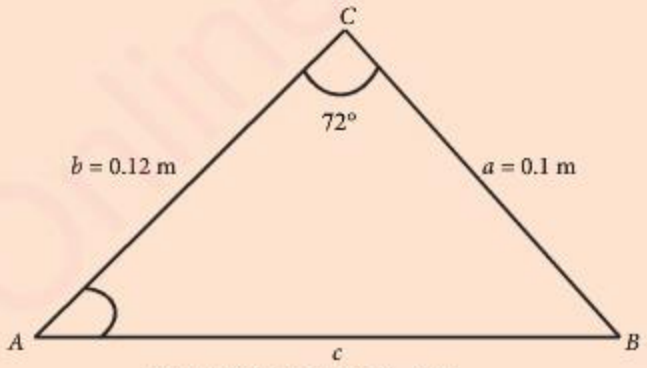

Figure 8.26 The triangle ABC

Using the cosine rule

$$c^2 = a^2 + b^2 - 2bc\cos C$$

$$c^2 = 0.1^2 + 0.12^2 - 2\times 0.1\times 0.12\times \cos 72°$$

$$\therefore \quad c = 0.13032$$

Hence, the length of the line AB is 0.13 m.

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Example 8.10: Find the size of angle C of triangle ABC in which $\overline{BC} = 16.4$ cm, $\overline{AC} = 11.2$ cm and $\overline{AB} = 23.57$ cm.

Solution: Consider Figure 8.27.

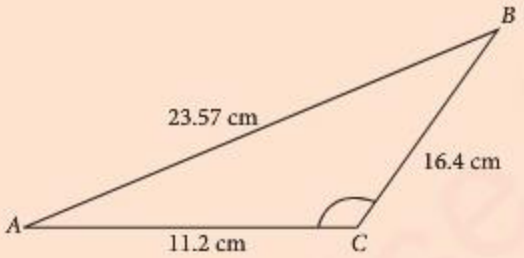

Figure 8.27 The triangle ABC

By using the Cosine rule,

$$c^2 = a^2 + b^2 - 2bc\cos C$$

$$23.57^2 = 16.4^2 + 11.2^2 - 16.4 \times 11.2 \times \cos C$$

$$555.5449 = 268.96 + 125.44 - 367.36\cos C$$

$$367.36\cos C = -161.1449$$

$$\cos C = \frac{-161.1449}{367.36}$$

$$\cos C = -0.4386$$

$$C = \cos^{-1}(-0.4386)$$

$$C = 116^\circ$$

$\therefore$ The size of angle C is 116°

Remark: We can apply cosine rule either when:–

i) two sides and an angle between them are given,
ii) all three sides of triangle are given.

Activity 8.3

How does the Pythagoras theorem relate to the cosine rule?

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Exercise 8.4

1. In triangle XYZ, $\overline{XY} = 0.3$ m, $\overline{YZ} = 0.2$ m and $\overline{ZX} = 0.15$ m. Find the smallest angle in the triangle.
2. Find the length of the side $\overline{QR}$ of triangle PQR where the lengths $\overline{PR} = 10.5$ cm, $\overline{PQ} = 18.8$ cm and $Q\hat{P}R = 11.7°$.
3. If a triangle has sides with lengths 0.06 m, 0.08 m, and 0.12 m, by how much does its largest angle differ from a right angle?
4. The sides m, n and p of a triangle MNP are in the ratio $4:9:7$. Find the smallest angle in the triangle.
5. A survey of a triangular field bounded by straight fences, has the three sides of length 200 m, 160 m and 130 m. Find correct to two decimal places the angles between the boundary fences.
6. A ship rounds a headland by sailing first 10 kilometres on a course of 072° then 14.5 kilometres on a course of 302°. Calculate correct to three decimal places the distance of its new position from its original position.

8.3 Trigonometric identities

a) The compound angle formulae

The compound angle formulae can be used to find the sums and differences of sine, cosine, and tangent of compound angles. Consider Figure 8.28.

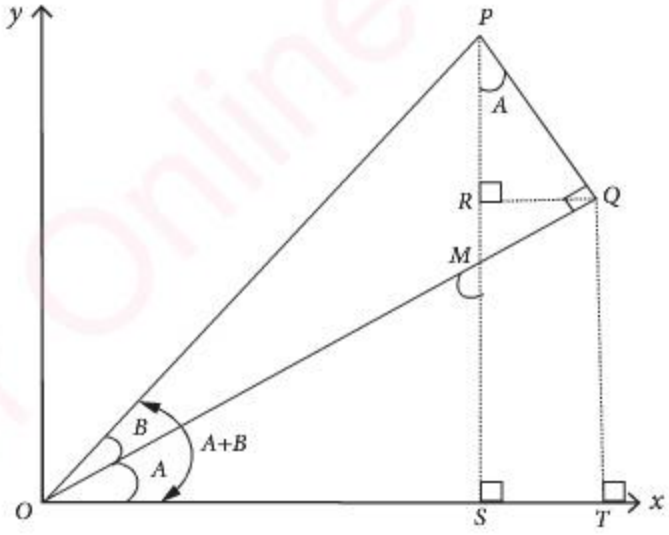

Figure 8.28 Illustration fo deriving compound angle formulae

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From triangle OSM

$$A+90°+O\hat{M}S=180° \quad (8.17)$$

$$O\hat{M}S=90°-A$$

For the triangle PQM

$$90°+A+P\hat{M}Q=180°$$

$$P\hat{Q}M=90°-A \quad (8.18)$$

Thus, equation (8.17) and (8.18) form opposite angles.

Let $\overline{OP}=r$

$$\sin B=\frac{\overline{PQ}}{\overline{OP}}$$

$\overline{PQ}=\overline{OP}\sin B$ but $\overline{OP}=r$, then

$$\overline{PQ}=r\sin B \quad (8.19)$$

$$\cos B=\frac{\overline{OQ}}{\overline{OP}}$$

$\overline{OQ}=\overline{OP}\cos B$ but $\overline{OP}=r$, then

$$\overline{OQ}=r\cos B \quad (8.20)$$

$$\sin A=\frac{\overline{QT}}{\overline{OQ}}$$

$\overline{QT}=\overline{OQ}\sin A$ but $\overline{OQ}=r\cos B$, then

$$\overline{QT}=r\cos B\sin A \quad (8.21)$$

$$\cos A=\frac{\overline{OT}}{\overline{OQ}}$$

$$\overline{OT}=\overline{OQ}\cos A$$

But, from equation (8.20), $\overline{OQ}=r\cos B$,

then

$\overline{OT}=\overline{OQ}\cos A$ which implies that;

$$\overline{OT}=r\cos A\cos B \quad 8.22)$$

Using triangle PQR, we have

$\sin A=\frac{\overline{RQ}}{\overline{PQ}}$ and $\cos A=\frac{\overline{PR}}{\overline{PQ}}$ which

leads to $\overline{RQ}=\overline{PQ}\sin A$ and

$$\overline{PR}=\overline{PQ}\cos A \quad (8.23)$$

Substituting Equation (8.19) into Equation (8.23), we obtain;

$\overline{RQ}=r\sin A\sin B$ and

$$\overline{PR}=r\sin B\cos A \quad (8.24)$$

Now, $\sin(A+B)=\frac{\overline{PS}}{\overline{OP}}$ but $\overline{OP}=r$

$$=\frac{\overline{PR}+\overline{RS}}{r} \text{ but } \overline{RS}=\overline{QT}$$

$$=\frac{\overline{PR}+\overline{OT}}{r}$$

From equation (8.21),

$$\overline{QT}=r\cos B\sin A$$

$$=\frac{r\sin B\cos A+r\cos B\sin A}{r}$$

$$=\sin B\cos A+\cos B\sin A$$

Therefore,

$$\sin(A+B)=\sin A\cos B+\cos A\sin B \quad (8.25)$$

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From Equation (8.25), replace angle B by $-B$;

Hence, $\sin\left(A+(-B)\right)=\sin A\cos(-B)+\cos A\sin(-B)$

But, $\cos(-B)=\cos B$ and $\sin(-B)=-\sin B$

$$\therefore \quad \sin(A-B)=\sin A\cos B-\cos A\sin B \tag{8.26}$$

From Figure 8.28

$$\cos(A+B)=\frac{\overline{OS}}{\overline{OP}}$$

$$=\frac{\overline{OT}-\overline{ST}}{\overline{OP}}$$

but, $\overline{ST}=\overline{RQ}$ and $\overline{OP}=r$

Therefore;

$\cos(A+B)=\dfrac{\overline{OT}-\overline{RQ}}{r}$, but from Equation (8.22) and (8.24), we have;

$OT = r\cos B\cos A$ and $\overline{RQ} = r\sin B\sin A$

Thus,

$$\cos(A+B)=\frac{r\cos B\cos A-\sin B\sin A}{r}$$

$$=\cos B\cos A-\sin B\sin A$$

$$\therefore \quad \cos(A+B)=\cos A\cos B-\sin A\sin B \tag{8.27}$$

Replace angle B by $-B$ in equation (8.27);

Thus, $\cos(A+(-B))=\cos A\cos(-B)-\sin A\sin(-B)$

$$\therefore \quad \cos(A-B)=\cos A\cos B+\sin A\sin B \tag{8.28}$$

Then, from the definition $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$,

We have,

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}$$

So,

$$\tan(A+B)=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B} \tag{8.29}$$

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Dividing the numerator and the denominator of the right hand side of the Equation (8.29) by cosAcosB leads to;

$$\tan(A+B) = \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}}$$

$$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B} \qquad (8.30)$$

Replace angle B by $-B$ in Equation (8.30),.

Hence, $\tan(A+(-B)) = \frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}$

But $\tan(-B) = -\tan B$

$$\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B} \qquad (8.31)$$

Summary of the Compound Angle Formulae

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
3. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
4. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
5. $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$
6. $\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$

Example 8.11: Find in surd form, the values of the following compound angles:

a) $\sin(30°+45°)$ b) $\cos(30°+45°)$ c) $\tan(30°+45°)$

Solution

a) $\sin(30°+45°) = \sin 30°\cos 45° + \cos 30°\sin 45°$

$$= \left(\frac{1}{2}\times\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$

$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$

$$\therefore \sin(30°+45°) = \frac{\sqrt{2}+\sqrt{6}}{4}$$

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b) From

$\cos(30^0 + 45^\circ) = \cos 30 \cos 45 - \sin 30 \sin 45$

$$= \left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2} \times \frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\therefore \cos(30^\circ + 45^\circ) = \frac{\sqrt{6} - \sqrt{2}}{4}$$

c) $\tan(30^\circ + 45^\circ) = \dfrac{\tan 30^\circ + \tan 45^\circ}{1 - \tan 30^\circ \tan 45^\circ}$

$$= \frac{\left(\frac{\sqrt{3}}{3} + 1\right)}{\left(1 - \left(\frac{1}{\sqrt{3}} \times 1\right)\right)}$$

$$= \frac{\left(\frac{\sqrt{3} + 3}{3}\right)}{\left(\frac{3 - \sqrt{3}}{3}\right)}$$

$$= \left(\frac{3 + \sqrt{3}}{3}\right) \times \left(\frac{3}{3 - \sqrt{3}}\right)$$

$$= \left(\frac{3 + \sqrt{3}}{3 - \sqrt{3}}\right)$$

$$= \left(\frac{3 + \sqrt{3}}{3 - \sqrt{3}}\right) \times \left(\frac{3 + \sqrt{3}}{3 + \sqrt{3}}\right)$$

By rationalizing denominator,

$$= \frac{3\sqrt{3} + 9 + 3 + 3\sqrt{3}}{9 - 3}$$

$$= \frac{12 + 6\sqrt{3}}{6}$$

$$= 2 + \sqrt{3}$$

$\therefore \tan(30^\circ + 45^\circ) = 2 + \sqrt{3}.$

Example 8.12: If $\tan 105^\circ = -2 - \sqrt{3}$, without using mathematical tables or calculators, find the value of $\tan 165$ in the form $a + b\sqrt{3}$ where a and b are integers.

Solution

$\tan 165^\circ = \tan(105^\circ + 60^\circ)$ gives

$$\tan 165^\circ = \frac{\tan 105^\circ + \tan 60^\circ}{1 - \tan 105^\circ \tan 60^\circ}$$

$$= \frac{\left(-2 - \sqrt{3}\right) + \sqrt{3}}{1 - (-2\sqrt{3})(\sqrt{3})}$$

$$= \frac{-2}{1 + 2\sqrt{3} + 3}$$

$$= \frac{2(-1)}{2(2 + \sqrt{3})}$$

$$= \frac{-1}{2 + \sqrt{3}}$$

$$= \left(\frac{-1}{2 + \sqrt{3}}\right) \times \left(\frac{2 - \sqrt{3}}{2 - \sqrt{3}}\right)$$

$$= \frac{-2 + \sqrt{3}}{4 - 3}$$

$\therefore \tan 165^\circ = -2 + \sqrt{3}$

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Deducing double angle formulae

To deduce the double angle formulae for sine, cosine, and tangent, the compound angle formulae will be involved. This is done when two angles are the same, i.e. $A = B$.

From

$\sin(A+B) = \sin A\cos B + \cos A\sin B,$

let $A = B$.

Then sin2A =

$\sin(A+A) = \sin A\cos A + \cos A\sin A$

$\therefore \sin 2A = 2\sin A\cos A$

which is the **double angle formula for sine**.

Similarly, from

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

we have

Cos2A =

$\cos(A+A) = \cos A\cos A - \sin A\sin A$

$\cos 2\ddot{u} = \cos^2 \quad -\sin^2$ (8.32)

Equation (8.32) is the **double angle formula for cosine**. This identity can be expressed in several forms as

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= (1-\sin^2 A) - \sin^2 A,$$
$$= 1 - 2\sin^2 A$$
$$\therefore \cos 2A = 1 - 2\sin^2 A.$$

Also, $\cos 2A = \cos^2 A - (1-\cos^2 A)$

$$= \cos^2 A - 1 + \cos^2 A$$
$$\therefore \cos 2A = 2\cos^2 A - 1.$$

Also, from $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$,

we have;

$\tan 2A = \tan(A+A) = \dfrac{\tan A + \tan A}{1 - \tan A\tan A}$

$$\therefore \tan 2A = \frac{2\tan A}{1-\tan^2 A} \qquad (8.33)$$

The equation (8.33) is the **double angle formula for tangent.**

Example 8.13: Prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Solution

Use $\sin 3\theta = \sin(2\theta + \theta)$. Then

$$\sin 3\theta = \sin 2\theta\cos\theta + \sin\theta\cos 2\theta$$
$$= (2\sin\theta\cos\theta)\cos\theta + \sin\theta(\cos^2\theta - \sin^2\theta)$$
$$= 2\cos\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta.$$

But $\cos^2\theta = 1 - \sin^2\theta$

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So $\sin 3\theta = 2\sin\theta(1-\sin^2\theta)+\sin\theta(1-\sin^2\theta)-\sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - \sin^3\theta - \sin^3\theta$

$\therefore \quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$. Hence proved.

Example 8.14: Find the value of θ such that $\sin 2\theta - \cos\theta = 0$ for $0° \le \theta \le 360°$

Solution

Given $\sin 2\theta - \cos\theta = 0$. Then

$2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

So, $\cos\theta = 0$ or $2\sin\theta - 1 = 0 \Rightarrow \cos\theta$ or $\sin\theta = \frac{1}{2}$.

In the interval $0 \le \theta \le 360°$;

$\cos\theta = 0$ implies that $\theta = 90°$ and $270°$; and $\sin\theta = \frac{1}{2}$ implies $\theta = 30°$ and $150°$

Therefore, the values of θ satisfying the equation $\sin 2\theta - \cos\theta = 0$, for the given interval are $\theta = 30°, 90°, 150°$ and $270°$.

Exercise 8.5

1. If $\sin A = \frac{4}{5}$ and $\cos B = \frac{7}{25}$, where A and B are acute angles, without using calculating devices, find the values of
   a) $\cos(A+B)$
   b) $\tan(A-B)$

2. If $\tan A = \frac{7}{24}$, where A is an angle in the third quadrant, without using calculating devices, evaluate each of the following:
   a) $\sin 2A$
   b) $\cos 2A$
   c) $\tan 2A$

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3. Without using calculating devices, find the exact values of each of the following:
   a) $\sin 75°$
   b) $\cos 65°\cos 35° + \sin 65°\sin 35°$
   c) $\dfrac{\tan 45° - \tan 150°}{1 + \tan 45°\tan 150°}$

4. Given that $\cos\theta = k$ and $0 \le \theta \le \dfrac{\pi}{2}$, find
   a) $\cos(\pi + \theta)$
   b) $\cos ec(\pi + \theta)$
   c) $\tan\left(\dfrac{\pi}{4} + \theta\right)$

5. Eliminate θ from each of the following pair of equations:
   a) $\begin{cases} x = \tan\theta \\ y = \tan 2\theta \end{cases}$
   b) $\begin{cases} x = \cos 2\theta - 1 \\ y = 2\sin\theta \end{cases}$
   c) $\begin{cases} x = \cos 2A \\ y = \cos^2 A \end{cases}$

6. Prove each of the following identities;
   a) $\dfrac{\cos 2\theta}{\cos\theta + \sin\theta} = \cos\theta - \sin\theta$
   b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
   c) $\dfrac{1 + \cos 2\alpha}{\sin 2\alpha} = \cot\alpha$

7. Solve the following equations:
   a) $3\cos 2\theta - \sin\theta + 2 = 0;\quad 0° \le \theta \le 360°$
   b) $\cos(\theta + 60°) = \sin 60°;\ 0° \le \theta \le 360°$
   c) $\tan 2\theta = 1;\ -\pi \le \theta \le \pi$
   d) $\cos 2\theta = \dfrac{1}{2};\ -\pi \le \theta \le \pi$

8.4 Graphs of the trigonometric functions

a) Sine function

The sine function is defined as $f(\theta) = \sin\theta$. In order to draw the graph of $f(\theta)$ prepare a table of values and then use it to draw the required graph.

Table 8.2. The table of values for $f(\theta) = \sin\theta$

θ	-2π	$\frac{-3\pi}{2}$	$-\pi$	$\frac{-\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f(\theta) = \sin\theta$	0	1	0	−1	0	1	0	−1	0

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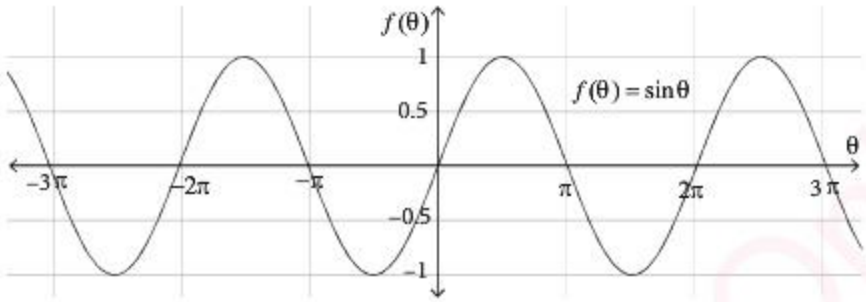

Figure 8.29 The graph of $f(\theta) = \sin\theta$.

b) Cosine function

The cosine function is defined as $f(\theta) = \cos\theta$. In order to draw the graph of $f(\theta)$ prepare a table of values for $f(\theta) = \cos\theta$ and then use it for plotting the curve.

Table 8.3 The table of values for $f(\theta) = \cos\theta$

θ	-2π	$\frac{-3\pi}{2}$	$-\pi$	$\frac{-\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f(\theta) = \cos\theta$	1	0	−1	0	1	0	−1	0	1

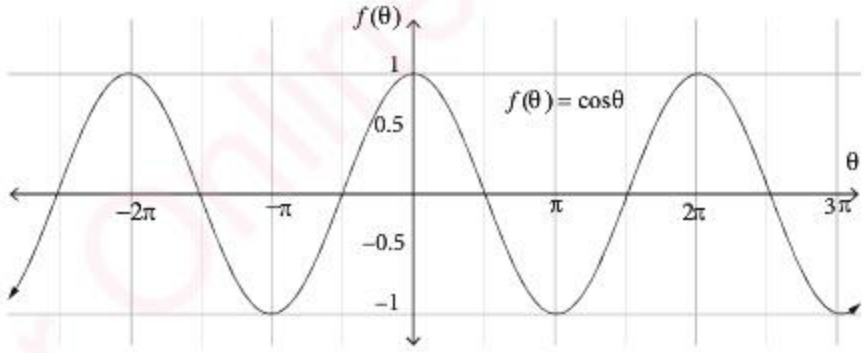

Figure 8.30 The graph of $f(\theta) = \cos\theta$

c) Tangent function

Tangent function is defined as $f(\theta) = \tan\theta$, and is undefined for some values of θ which are referred to as asymptotes. In order to draw the graph of $f(\theta)$ prepare a table of values and then use it to draw the graph.

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Table 8.4 The table of values for $f(\theta) = \tan\theta$

θ	-2π	$\frac{-3\pi}{2}$	$\frac{-5\pi}{4}$	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	2π
$f(\theta) = \tan\theta$	0	$-\infty$	-1	0	1	$-\infty$	-1	0	1	∞	-1	0	1	∞	0

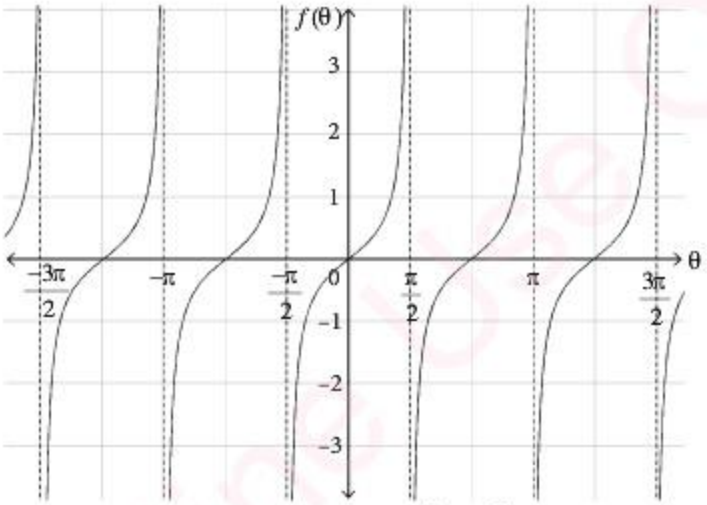

Figure 8.31 The graph of $f(\theta) = \tan\theta$

Properties of the graphs of trigonometric functions

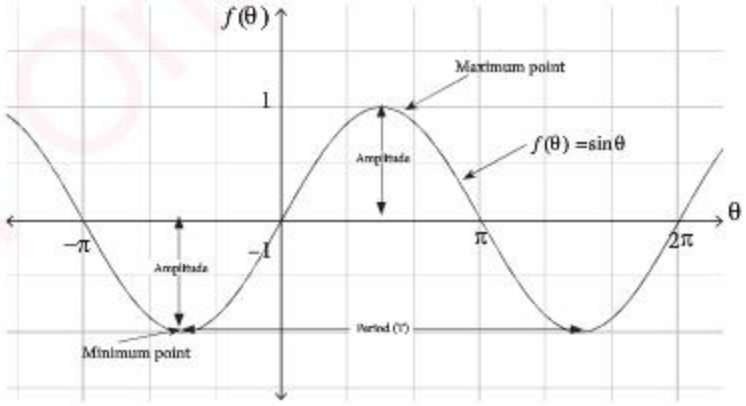

Figure 8.32 Graphical illustration on the properties of the sine function

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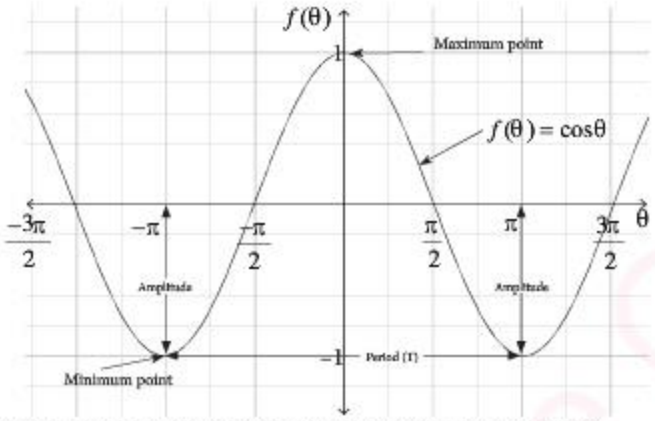

Figure 8.33 Graphical illustration on the properties of the cosine function

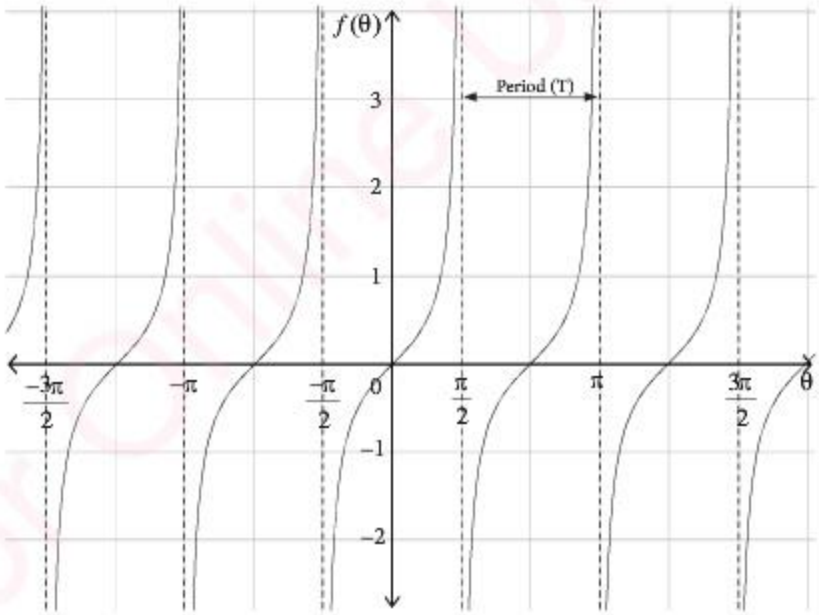

Figure 8.34 Graphical illustration on the properties of the tangent function

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1. Maximum and minimum

Maximum and minimum values of $f(x)=\sin\theta$ and $f(x)=\cos\theta$ functions are 1 and -1, respectively. However, the tangent function $f(\theta)=\tan\theta$ has neither maximum nor minimum values (see Figure 8.32, Figure 8.33 and Figure 8.34).

2. Periodic function

A function f is periodic with period T if $f(\theta+T)=f(\theta)$. Sine, cosine and tangent functions are periodic functions (i.e graphs repeat after every periodic angle T). Observations from their graphs show that sine and cosine functions have the period of 2π radians or 360° (see Figure 8.32 and Figure 8.33). That is,

$$\sin(\theta+2\pi n)=\sin\theta, \text{ and}$$
$$\cos(\theta+2\pi n)=\cos\theta,$$

where $n=0,\pm1,\pm2,\pm3,...$

While tangent function has the period $T=\pi$ radians or 180° (see Figure 8.34). That is,

$\tan(\theta+\pi n)=\tan\theta$ where

$n=0,\ \pm1,\ \pm2,\ \pm3,...$

3. Even and odd function

a) A function $f(\theta)$ is said to be an even function if $f(-\theta)=f(\theta)$. From Figure 8.32, $\cos(-\pi)=-1$ and $\cos(\pi)=-1$.

Therefore; $\cos(-\pi)=\cos(\pi)$.

This implies that cosine function is an even function.

b) A function $f(\theta)$ is said to be an odd function if $f(-\theta)=f(\theta)$. From Figure 8.33 and Figure 8.34, we observe that

$$\sin\left(\frac{\pi}{2}\right)=1 \text{ and } \sin\left(-\frac{\pi}{2}\right)=-1,$$
$$\therefore \sin\left(-\frac{\pi}{2}\right)=-\sin\left(\frac{\pi}{2}\right),$$

Similarly, $\tan\left(-\frac{\pi}{4}\right)=-1$ and $\tan\left(\frac{\pi}{4}\right)=-1$

$$\therefore \tan\left(-\frac{\pi}{4}\right)=-\tan\left(\frac{\pi}{4}\right)$$

Therefore, sine and tangent functions are odd functions

i.e. $\sin(-\theta)=-\sin(\theta)$ and

$\tan(-\theta)=-\tan(\theta)$

4. Amplitude

This is the highest value in which sine and cosine function graphs attain. Tangent graph has no amplitude. See Figure 8.32 and Figure 8.33. If $f(\theta)=a\sin\theta$ or $f(\theta)=a\cos\theta$ then the amplitude of the sine or cosine function is $|a|$.

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Exercise 8.6

1. Identify features of graphs of sine, cosine, and tangent of angles.
2. What is the relationship between graphs of sine and cosine functions?
3. Sketch $f(t)=3\sin 2t$ then, find the amplitude and period of the function.
4. By giving the reason in each case, show whether or not the following are even or odd functions:
   a) $f(x)=\sin 4x$
   b) $f(x)=\tan 2x$
   c) $f(x)=\cos(1+3x)$
5. Sketch the graphs in the interval $-360^\circ \le x \le 360^\circ$ and state the amplitude and the period in each of the following:
   a) $f(\theta)=2\sin 3\theta$
   b) $f(\theta)=1+\cos\theta$

8.5 Calculus of trigonometric functions

a) *The derivatives of trigonometric functions*

Example 8.15: Find the derivative of $f(x)=\sin x$ from the first principle.

Solution: Given $f(x)=\sin x$ and $f(x+h)=\sin(x+h)$

Using the first principle of differentiation

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$

$$=\lim_{h\to 0}\left(\frac{\sin x(\cos h-1)}{h}+\frac{\cos x\sin h}{h}\right)$$

As $h\to 0$, then $\frac{\cos h-1}{h}\to 0$ and $\frac{\sin h}{h}\to 1$.

This can be shown numerically in Table 8.5

Table 8.5 Values of sine

h	$\frac{\sin h}{h}$	$\frac{\cos h-1}{h}$
±1	0.8414098	0.459884
±0.5	0.95885108	0.24483485
±0.4	0.9354586	0.1934515
±0.3	0.985056	0.1488336
±0.2	0.993346	0.0996 10
±0.1	0.99833417	0.049958347
±0.05	0.99958339	0.02499492
±0.01	0.99998333	0.004999958
±0.005	0.99999583	0.002500000
±0.001	0.99999983	0.000500000

$f'(x)=(\sin x)(0)+(\cos x)(1)=\cos x$

Therefore, $\frac{d}{dx}(\sin x)=\cos x$

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Example 8.16: Differentiate $g(x)=\cos x$ using the first principle.

Solution: Let $g(x)=\cos x$,

$$g(x+h)=\cos(x+h)$$

Using the first principle of differentiation

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$g'(x)=\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=\lim_{h\to 0}\left(\frac{\cos x(\cos h-1)}{h}-\frac{\sin x\sin h}{h}\right)$$

As $h\to 0$, then $\frac{\sin h}{h}\to 1$ and $\frac{\cos h-1}{h}\to 0$

Then,

$$g'(x)=(\cos x)(0)-(\sin x)(1)=-\sin x$$

Therefore, $\frac{d}{dx}(\cos x)=-\sin x$

Example 8.17: Use the quotient rule to show that $\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$

Solution

Let $y=\cot x=\frac{\cos x}{\sin x}$.

Let $u=\cos x\Rightarrow\frac{du}{dx}=-\sin x$,

$$v=\sin x=\frac{dv}{dx}=\cos x$$

So $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$$=\frac{(\sin x)(-\sin x)-(\cos x)(\cos x)}{\sin^2 x}$$

$$=\frac{-\sin^2 x-\cos^2 x}{\sin^2 x}$$

$$=\frac{-1}{\sin^2 x}=-\left(\frac{1}{\sin x}\right)^2=-\text{cosec}2x$$

Therefore, $\frac{d}{dx}(\cot x)=-\cos ec^2 x$

Example 8.18: Find the derivative of $y=\sin(2x+3)$

Solution: From $y=\sin(2x+3)$

Let $u=2x+3$,

$$\frac{du}{dx}=2$$

So, $y=\sin u$

$$\frac{dy}{du}=\cos u$$

By using chain rule

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$\frac{dy}{dx}=2\cos u$$

$$\therefore\quad\frac{dy}{dx}=2\cos(2x+3)$$

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Example 8.19: Find the derivative of $y = \cos^2 x$

Solution: From $y = \cos^2 x$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

So, $y = u^2$

$\frac{dy}{du} = 2u$

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = -2\cos x \sin x = -\sin 2x$$

Example 8.20: Differentiate $y = \cos x \sin x$

Solution: Given $y = \cos x \sin x$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

$v = \sin x$, $\frac{dv}{dx} = \cos x$

So $y = uv$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

By product law, we have;

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = -\sin^2 x + \cos^2 x$$

$$\therefore \frac{dy}{dx} = \cos^2 x - \sin^2 x = \cos 2x$$

Exercise 8.7

1. Differentiate $f(x) = \sin 3x$ using the first principle.
2. Find the derivative of $f(\theta) = 1 + \cos 2\theta$ using the first principle.
3. Show that $\frac{d}{dx}(\tan x) = \sec^2 x$ using the following methods:
   a) The first principle
   b) The quotient rule
4. Show that $\frac{d}{dx}(\operatorname{cosec}\theta) = -\operatorname{cosec}\theta \cot\theta$ by using the first principle and quotient rule.
5. Use both the first principle and the quotient rule to find $\frac{d}{dx}(\sec\theta)$
6. Find $\frac{dy}{dx}$ given that
   a) $y = \tan 2x$
   b) $y = \sec(1 + 5x)$
7. Find the following:
   a) $\frac{d}{dx}\left(\sin\left(x^2 + 4x + 5\right)\right)$
   b) $\frac{d}{dx}\left(\sqrt{1 + \sin x}\right)$

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8. Show that

a) $\frac{d}{d\theta}(\sin\theta\cos^3\theta) = 4\cos^4\theta - 3\cos^2\theta$

b) $\frac{d}{d\theta}(\sin\theta\tan 2\theta) = 2\sin\theta\sec^2 2\theta + \cos\theta\tan 2\theta$

9. Find the derivative of each of the following:

a) $y = \cos^5 4x$ b) $y = \cot^2 3\theta$

10. Differentiate each of the following with respect to θ :

a) $y = \frac{\cos\theta}{1+\sin\theta}$ b) $y = \frac{\sin 3\theta}{\cos 2\theta}$

b) The integration of trigonometric functions

Since integration is the reverse process of differentiation, then the following are standard integrals for trigonometric functions:

1. $\int \cos x\,dx = \sin x + c$
2. $\int \sin x\,dx = \cos x + c$
3. $\int \sec^2 dx = \tan x + c$
4. $\int \sec x\tan x\,dx = \sec x + c$
5. $\int \text{cosec}\, x\cot x\,dx = \text{cosec}\, x + c$
6. $\int \text{cosec}^2 x\,dx = -\cot x + c$

where c is a constant of integration.

Example 8.21: Find the $\int \sin(6x-5)dx$

Solution: Let $u = 6x-5$

$du = 6dx$

$dx = \frac{1}{6}du$

$\int \sin(6x-5)dx = \int \frac{\sin u}{6}du$

$= \frac{1}{6}\int \sin u\,du$

$= \frac{1}{6}(-\cos u) + c$

But $u = 6x-5$

$\therefore \int \sin(6x-5)dx = -\frac{1}{6}\cos(6x-5) + c$

Example 8.22: Show that

$\int_0^{\frac{\pi}{4}} \cos^2 x dx = \frac{\pi}{8} + \frac{1}{4}$

Solution

$\int_0^{\frac{\pi}{4}} \cos^2 x\,dx = \int_0^{\frac{\pi}{4}} \frac{1+\cos 2x}{2}dx,$

From $\cos 2x = 2\cos^2 x - 1$

$= \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} dx + \int_0^{\frac{\pi}{4}} \cos 2x dx\right]$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

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$$=\frac{1}{2}\left[\left(\frac{\pi}{4}+\frac{1}{2}\sin\frac{\pi}{2}\right)-(0+0)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{4}+\frac{1}{2}\right]=\frac{\pi}{8}+\frac{1}{4}$$

$\therefore \int_0^{\frac{\pi}{4}} \cos^2 x dx = \frac{\pi}{8}+\frac{1}{4}$ Hence shown.

Example 8.23: Find $\int \sec^2(3x+8)dx$

Solution:

Given $\int \sec^2(3x+8)dx$.

let $u = 3x+8$

$$\frac{du}{dx}=3 \Rightarrow dx=\frac{1}{3}du$$

$$\int \sec^2(3x+8)\,dx = \frac{1}{3}\int \sec^2 u\,du$$

$$=\frac{1}{3}\tan u + c$$

$$=\frac{1}{3}\tan(3x+8)+c$$

$\therefore \int \sec^2(6x-5)dx = \frac{1}{3}\tan(3x+8)+c$

Exercise 8.8

1. Find the following:
   a) $\int \sin(7x+4)dx$
   b) $\int \cos^3\theta\, d\theta$
   c) $\int \frac{\cos^2\theta}{\sin^2 2\theta} d\theta$
   d) $\int 5\cos(2x-7)dx$

2. Evaluate the following:
   a) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx$
   b) $\int_0^{\frac{\pi}{6}} \frac{\sin x}{\cos^3 x} dx$
   c) $\int_0^{\frac{\pi}{4}} 2\sin 4\theta\ d\theta$
   d) $\int_0^{\frac{\pi}{8}} \cos(1+2x)\ dx$

3. Show that
   $\int \cos x \sin^5 x\,dx = \frac{1}{6}\sin 6x + c.$

4. Express $\tan^2\theta$ in terms of $\sec^2\theta$, then show that $\int \tan^2\theta\ d\theta = \tan\theta - \theta + c.$
   Hence, evaluate correct to 4 decimal places $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \tan^2\theta\, d\theta.$

5. Verify that
   $\int \sin^3 x\,dx = \frac{1}{3}\cos^3 x - \cos x + c.$

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Revision exercise 8

1. Eliminate θ from each of the following pair of equations:

   a) $\begin{cases} x = tan\,2\theta \\ y = tan\,\theta \end{cases}$ b) $\begin{cases} x = 1 + cos\,2\theta \\ y = 4\,sin\theta \end{cases}$

2. Prove each of the following identities:

   a) $\sec 2\theta + \tan 2\theta = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

   b) $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

   c) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

   d) $\dfrac{1-2\sin^2 A}{2\sin A\cos A} = \cot 2A$

3) Simplify each of the following expressions:

   a) $\dfrac{1-2\sin^2\theta}{\sin 2\theta}$ b) $cos\,58°\,cos\,32° - sin\,58°\,sin\,32°$ c) $\dfrac{\tan 4\theta + \tan 2\theta}{1 - \tan 4\theta\,\tan 2\theta}$

4) Solve the following equations

   a) $8\,sin\,\theta\,cos\,\theta = 3;\ for\ -180° \le \theta \le 180°$

   b) $cos\,2\theta = cos\,\theta;\ for\ 0 \le \theta \le 2\pi$

   c) $sin\,2\theta + sin\,\theta = 0;\ for\ -\pi \le \theta \le \pi$

5. a) In triangle JKL, $\overline{JK} = 0.06$ m, $\overline{KL} = 0.09$ m, and $K\hat{J}L = 52°$, find the value of $J\hat{L}K$.

   b) In the triangle STU, calculate the size of $S\hat{T}U$.

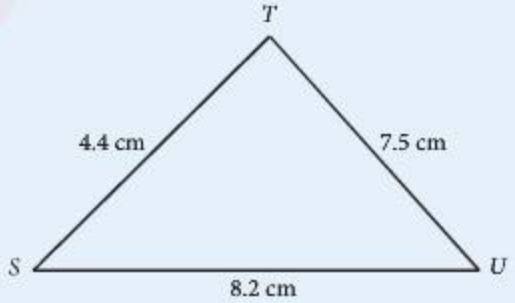

Figure 8.35 The triangle STU

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6. Draw the graph of each of the following:

   a) $f(\theta) = \sin\theta$, for $0^0 \le \theta \le 180^0$

   b) $h(\theta) = \tan\theta$; for $-\pi \le \theta \le \pi$

7. Differentiate each of the following with respect to x:

   a) $y = \cos^3\theta$

   b) $y = \tan\frac{3}{2}x$

   c) $y = \frac{\sin 2x}{\cos 5x}$

   d) $y = \sin^2 x \tan 3x$

8. Differentiate each of the following using the first principle:

   a) $f(x) = \cos 3x$

   b) $f(\theta) = \sin\frac{1}{3}\theta$

9. Find each of the following:

   a) $\int 2\sin\left(\frac{1}{2}x+3\right)dx$

   b) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

   c) $\int \frac{\cos\theta}{\sin^3\theta}\,d\theta$

   d) $\int \frac{1}{2}\cos ec^3 x \cot x dx$

10. Evaluate each of the following correct to three decimal places:

    a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 3x\,dx$

    b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x\,dx$

    c) $\int_0^{\frac{\pi}{4}} \sin^2\theta\,d\theta$

    d) $\int_0^{\frac{\pi}{8}} \cos 4x \sin^3 4x\,dx$

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Chapter Nine

Exponential and logarithmic functions

Introduction

Exponential functions are the functions of the form $f(x)=a^x$, where $a>0$; while logarithmic functions are the inverses of exponential functions, which are written in the form $f(x)=\log_a x$, where the base $a>0$. Knowledge of exponential functions can be applied in various real life problems such as in determining compound interest and depreciation, population growth, population decay, radioactive decay and Newton's law of cooling. In this chapter, you will learn about the basic properties of exponential functions, calculus of the exponential functions and basic properties of logarithmic functions, find derivatives of logarithmic and exponential functions, integration of rational functions and some real life applications of exponential and logarithmic functions

9.1 Basic properties of exponential functions

Exponential function is written in the form of $f(x)=a^x$ whereby a is a constant base of the function and x is a variable called exponent.

Base of the function is any positive real number and the most frequently used bases are base 10 that is, $f(x)=10^x$ and base e, that is, $f(x)=e^x$ whereby e is a Naperian constant whose numerical value is approximately equal to 2.71828.

Laws of exponential functions

a) If $a>0$ then $a^x>0$

b) $a^{x+y}=a^x a^y$ or $a^x\times a^y$

c) $a^{x-y}=\dfrac{a^x}{a^y}$

d) $(ab)^x=a^x b^x=a^x\times b^x$

e) $(a^x)^y=a^{xy}$

f) $\dfrac{a^x}{a^x}=1$. But also $\dfrac{a^x}{a^x}=a^{x-x}=a^0$

Hence $a^0=1$

g) $a^{-x}=\dfrac{a^0}{a^x}=\dfrac{1}{a^x}$

Example 9.1: Identify the base and exponent of the following:

a) $f(x)=e^{2x+3}$ b) $f(x)=\left(\dfrac{1}{2}\right)^x$

Solution

a) From $f(x)=e^{2x+3}$

$\therefore$ Base $=e$ and exponent $=2x+3$

b) From $f(x)=\left(\dfrac{1}{2}\right)^x$

$\therefore$ Base $=\dfrac{1}{2}$ and exponent $=x$

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Example 9.2: Solve for x from the following exponential equation

$3^{x^2-5x+2} = 9^{x+1}$

Solution

From $3^{x^2-5x+2} = 9^{x+1}$

$3^{x^2-5x+2} = 3^{2(x+1)}$

From the basic properties of exponents, that is, if $a^y = a^b$ then $y = b$ then,

$x^2 - 5x + 2 = 2(x+1)$

$x^2 - 5x + 2 = 2x + 2$

$x^2 - 7x = 0$

$x(x-7) = 0$

$x = 0$ or $x = 7$

Exercise 9.1

1. Write the base and exponent of each of the following:

   a) $f(x) = 3^x$ b) $f(x) = b^{x+2}$

   c) $f(x) = e^{2x+3}$

2. Express each of the following as a single base.

   a) $\left(\frac{p}{q^4}\right)^3 \left(pq^3\right)^4$ b) $(ab)^8 \left(\frac{a}{b^4}\right)^2$

   c) $x^k x^{k+2} x^{k+3}$

3. Evaluate each of the following expression if $a = -4$ and $k = 2$

   a) $\left(a^3\right)^k \left(a^k\right)^2 a^{2k-1}$

   b) $a^k + a^{2k-1} - a^{3k+1}$

   c) $\left(\left(a^k\right)^2 \left(a^2\right)^k\right) \div a^{2k+3}$

4. Use properties of exponents to simplify each of the following expressions:

   a) $\left(a^y + 2^{-y}\right)^2 - \left(2^y - 2^{-y}\right)^2$

   b) $\frac{p^{3q+2}\left(p^{q-1}\right)^2}{\left(p^{q+1}\right)^3}$

   c) $\frac{x^{3p+q} x^{3q+p}}{\left(x^p\right)^2}$

5. Determine the value of x in the following equations

   a) $8^x = \frac{1}{16^{x+1}}$

   b) $3^{2x} - 3^x - 6 = 0$

   c) $2^{\left(x^2+3x-2\right)} = 4^x$

Graphs of exponential functions

In drawing the graphs of exponential function $f(x) = a^x$, when $a > 1$, the function $f(x)$ increases as x increases (grows exponentially) and when $0 < a < 1$, the function $f(x)$ decreases (decays exponentially) as x increases (see Figures 9.1 and 9.2).

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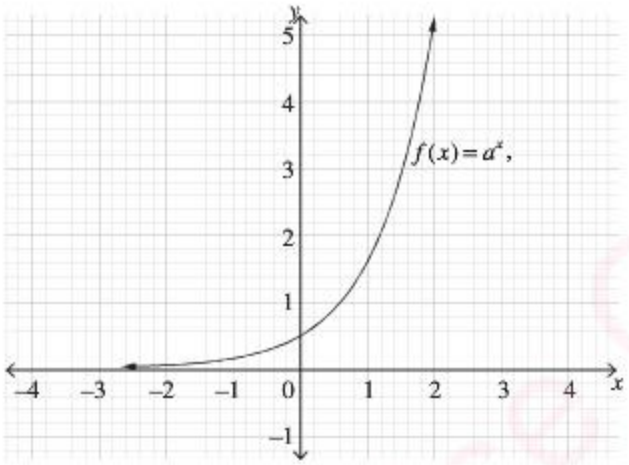

Figure 9.1 Graph of $f(x)=a^x$, when $a>1$

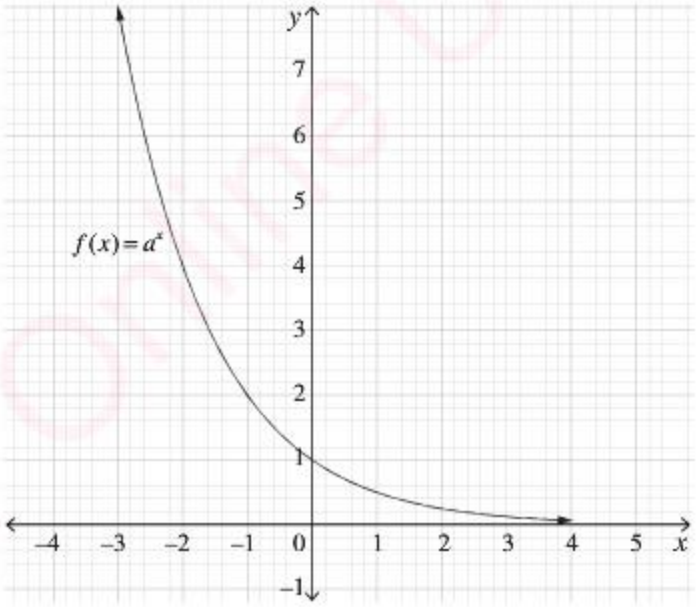

Figure 9.2: Graph of $f(x)=a^x$, when $0<a<1$

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From Figure 9.1 and Figure 9.2 the graph of $f(x)=a^x$, when $a>0$ and $f(x)=a^x$, when $0<a<1$, respectively contain point (0,1), since $a^0=1$ which is y-intercept. There is no x-intercept since the graph does not cross the x – axis. Moreover, as x approaches $-\infty$ or $+\infty$ in Figure 9.1 and Figure 9.2, respectively both graphs approaches the x – axis but do not cross or touch it, as the x – axis (the line $y=0$) is the horizontal asymptote of the curve.

Example 9.3: Draw on the same axes the graph of:

a) $f(x)=3^x$ b) $g(x)=\left(\frac{1}{3}\right)^x$

Solution The table of values for $f(x)=3^x$ and $g(x)=\left(\frac{1}{3}\right)^x$

Table 9.1

x	–2	–1	0	1	2
$f(x)=3^x$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$g(x)=\left(\frac{1}{3}\right)^x$	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$

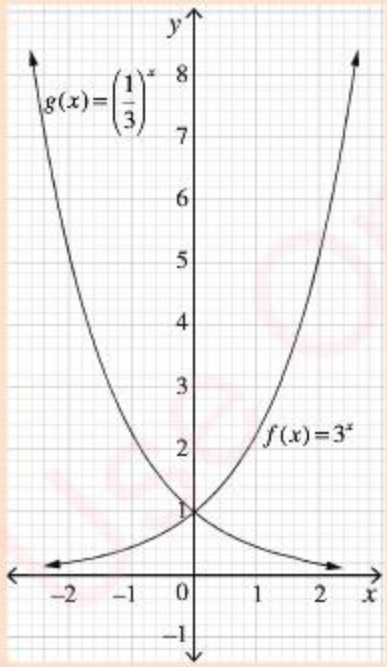

Figure 9.3 Exponential graph of $f(x)$ and $g(x)$

Domain and range of exponential functions

Any exponential function has a property that the input values are the domain and the output values are the range. Domain refers to the set of all real numbers while range is the set of all positive real numbers. That is the domain of a^x is $\{x : x \in \mathbb{R}\}$ and the range of a^x is $\{y : y>0\}$.

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Example 9.4: State the domain and range of the following:

a) $f(x)=3^x$

b) $g(x)=\left(\frac{1}{3}\right)^x$

Solution

From the graphs drawn in Figure 9.3;

The domain of

a) $f(x)=3^x$ is $\{x: x\in\mathbb{R}\}$

b) $g(x)=\left(\frac{1}{3}\right)^x$ is $\{x: x\in\mathbb{R}\}$

The range of

a) $f(x)=3^x$ is $\{f(x): f(x)>0\}$

b) $g(x)=\left(\frac{1}{3}\right)^x$ is $\{g(x): g(x)>0\}$

Exercise 9.2

1. Given $f(x)=2^x$ and $g(x)=4^{-x}$
   a) Construct the table of values for 2^x and 4^{-x}.
   b) Using the table of values in a) plot the graph of $f(x)$ and $g(x)$ on the same $xy-$plane.
   c) From the graph, state the domain and range of $f(x)$ and $g(x)$.
2. Draw the graphs of $f(x)=2^x$ and $g(x)=\left(\frac{1}{2}\right)^x$ on the same xy-plane.
3. Draw the graph of $f(x)=5^{1-x}$ and state its domain and range.
4. Draw the graph of $g(x)=3\left(3^x\right)$ in the interval $-2\le x\le 1$ and state its domain and range.
5. Draw the graph of $h(x)=-\left(\frac{1}{2}\right)^x$ and state its domain and range.

9.2 Calculus of the exponential function

a) Differentiation of exponential functions

The derivatives of $f(x)=e^x$ and $g(x)=a^x$ have a minor difference. Let us differentiate them by using the first principle and see how they differ.

Derivative of the exponential function $f(x)=e^x$

From the first principle:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$=\lim_{h\to 0}\frac{e^x e^h-e^x}{h}$$

$$=\lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h}$$

As $h\to 0$, then the term $\frac{e^h-1}{h}\approx 1$ (observe row 2 of Table 9.2)

Now, $f'(x)=e^x(1)$

Therefore, $f'(x)=e^x$

Derivative of the exponential function $g(x)=a^x$

From the first principle:

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$=\lim_{h\to 0}\frac{a^x a^h-a^x}{h}$$

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$$= \lim_{h \to 0} \frac{a^x\left(a^h - 1\right)}{h}$$

As $h \to 0$, the term $\frac{a^h - 1}{h} \approx k$, where k is a constant which depends on the value of a.

For example, as shown in Table 9.2, when:

$a = 2$, $\frac{2^h - 1}{h} \to 0.693$,

$a = 3$, $\frac{3^h - 1}{h} \to 1.386$, and

$a = 10$, $\frac{10^h - 1}{h} \to 2.3$.

Therefore, $g'(x) = ka^x$

Table 9.2 Values of some exponential functions expressed in terms of h as $h \to 0$

$h \to 0$	1	0.1	0.01	0.001	0.0001	0.00001
$\frac{e^h - 1}{h}$	1.71828	1.05171	1.0050	1.00050	1.00005	1.000005
$\frac{2^h - 1}{h}$	1	0.7177	0.6956	0.6934	0.69317	0.69315
$\frac{4^h - 1}{h}$	3	1.48698	1.39595	1.3873	1.3864	1.3863
$\frac{10^h - 1}{h}$	9	2.5893	2.3293	2.3052	2.3029	2.30261

Thus, from the Table 9.2, it can be concluded that $\frac{e^h - 1}{h} \approx 1$, $\frac{a^h - 1}{h} \approx k$ and $\frac{a^h - 1}{h} \approx 1$.

Therefore, the derivative of the exponential function $f(x) = e^x$ is given by $\frac{dy}{dx} = \frac{d}{dx}\left(e^x\right) = e^x$ and for $g(x) = a^x$ is $g'(x) = ka^x$, where k is a constant which depends on the value of a. This concept will be discussed further in subtopic 9.4.

Derivative of the exponential function $f(x) = e^{g(x)}$

To obtain the derivative of the function $y = e^{g(x)}$ where $g(x)$ is another function of x, apply chain rule of the differentiation as follows:

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}\frac{d}{dx}\left(g(x)\right)$$

Example 9.5: Find the derivative of the following:

a) $f(x) = e^{2x+3}$ b) $g(x) = e^{1-x^2}$

c) $h(x) = e^{3x}$ d) $g(x) = e^{3-2x}$

Solution

The first derivatives of the following exponential functions are obtained as follows:

a) $f'(x) = \frac{d}{dx}\left(e^{2x+3}\right)$

$= \left(e^{2x+3}\right)\frac{d}{dx}\left(2x+3\right)$

$= 2e^{2x+3}$

$\therefore f'(x) = 2e^{2x+3}$

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b) $g'(x)=\frac{d}{dx}\left(e^{1-x^2}\right)$

$=\left(e^{1-x^2}\right)\frac{d}{dx}\left(1-x^2\right)$

$=-2xe^{1-x^2}$

$\therefore \quad g'(x)=-2xe^{1-x^2}$

c) $h'(x)=\frac{d}{dx}\left(e^{3x}\right)$

$=\left(e^{3x}\right)\frac{d}{dx}\left(3x\right)$

$\therefore \quad h'(x)=3e^{3x}$

d) $g'(x)=\frac{d}{dx}\left(e^{3-2x}\right)$

$=\left(e^{3-2x}\right)\frac{d}{dx}\left(3-2x\right)$

$=-2e^{3-2x}$

$\therefore \quad g'(x)=-2e^{3-2x}$

Exercise 9.3

1. Find the derivative of the following:
   a) $f(x)=e^{\sqrt{x}}$ b) $h(x)=e^{3x-4}$
   c) $g(x)=e^{\sqrt{x}+3x^2}$
2. Determine the derivative of the following functions at the given value of x:
   a) $f(x)=x^2e^{4x}$ at $x=\frac{1}{4}$
   b) $g(x)=e^{x+\sqrt{x}}$ at $x=4$
   c) $h(x)=xe^{x^2}$ at $x=-\frac{1}{5}$
3. Find the second derivative of the following functions:
   a) $y=e^{ax+b}$ b) $y=3e^{1+x^2}$
   c) $y=e^{\sin x}$
4. If $y=e^{2t}$ and $x=4e^{-3t}$, find $\frac{dy}{dx}$ in terms of t.
5. If $x^2y-xe^{4y}=4$, show that $\frac{dy}{dx}=\frac{e^{4y}-2xy}{x^2-4xe^{4y}}$.

b) *Integration of the exponential function* $f(x)=e^x$

Recall the derivative of e^x

$$\frac{d}{dx}(e^x)=e^x$$

Integrate both sides with respect to x

$$\int\frac{d}{dx}\left(e^x\right)dx=\int e^x\,dx$$

$$=e^x+c$$

$$\therefore \quad \int e^x\,dx=e^x+c$$

Also, substitution technique can be used to find integrals of the forms: $\int e^{ax+b}\,dx$ or $\int f'(x)e^{f(x)}\,dx$

Example 9.6: Find:

a) $\int e^{3x+2}\,dx$ b) $\int e^{6-5x}\,dx$

c) $\int e^{\sin x}\cos x\,dx$ d) $\int 2xe^{6x^2+7}\,dx$

Solution:

a) $\int e^{3x+2}\,dx$

Let $u=3x+2$,

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$$\frac{du}{dx}=3$$

$$dx=\frac{du}{3}$$

Now $\int e^{3x+2}dx=\int e^{u}\frac{du}{3}$

$$=\frac{1}{3}\int e^{u}\,du$$

$$=\frac{1}{3}e^{u}+c,$$

but $u=3x+2$

$$\therefore \int e^{3x+2}\,dx=\frac{1}{3}e^{3x+2}+c$$

b) $\int e^{6-5x}dx$

Let $u=6-5x$

$$\frac{du}{dx}=-5$$

$$dx=\frac{du}{-5}$$

Now $\int e^{6-5x}dx=\int e^{u}\frac{du}{-5}$

$$=\frac{-1}{5}\int e^{u}\,du$$

$$=\frac{-1}{5}e^{u}+c$$

But $u=6-5x$

$$\therefore \int e^{3x+2}\,dx=\frac{-1}{5}e^{6-5x}+c$$

c) $\int e^{\sin x}\cos x\,dx$

Let $u=\sin x$

$$\frac{du}{dx}=\cos x$$

$$dx=\frac{du}{\cos x}$$

Now $\int e^{\sin x}\cos x\,dx=\int e^{u}\cos x\frac{du}{\cos x}$

$$=\int e^{u}du$$

$$=e^{u}+c$$

But $u=\sin x$

$$\therefore \int e^{\sin x}\cos x\,dx=e^{\sin x}+c$$

d) $\int 2xe^{6x^2+7u}dx$

Let $u=6x^2+7$

$$\frac{du}{dx}=12x$$

$$dx=\frac{du}{12x}$$

Now $\int 2xe^{6x^2+7}dx=\int e^{u}2x\frac{du}{12x}$

$$=\int e^{u}\frac{du}{6}$$

$$=\frac{1}{6}\int e^{u}\,du$$

$$=\frac{1}{6}e^{u}+c$$

But $u=6x^2+7$

$$\therefore \int 2xe^{6x^2}\,dx=\frac{1}{6}e^{6x^2+7}+c$$

Generally, if $f(x)$ is a linear function (i.e. $f(x)=ax+b$), then

$$\int e^{f(x)}dx=\frac{e^{f(x)}}{f'(x)}+c$$

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Exercise 9.4

1. Find the following :
   a) $\int e^{5x}\,dx$ b) $\int e^{-9x+1}dx$
   c) $\int e^{6+3x}\,dx$
2. Evaluate the following definite integrals:
   a) $\int_0^1 e^{-10x}dx$ b) $\int_0^{\frac{\pi}{2}} \sin x e^{\cos x}dx$
   c) $\int_0^1 4xe^{2x^2+4}\,dx$
3. Use the substitution technique to find
   $$\int (x+2)e^{x^2+4x+3}dx$$
4. Evaluate the following definite integral $\int_{-2}^{3} 3^x dx$.
5. Show that
   $$\int \left(e^x+e^{-x}\right)^2\left(e^{x^3+4x+3}\right)dx=\frac{\left(e^x+e^{-x}\right)^3}{3}+c$$

9.3 Basic properties of logarithmic functions

Consider an exponential function defined by $f(x)=2^x$. The inverse of $f(x)=2^x$ is $g(x)=\log_2 x$, the resulting inverse is a new function which defines relationship between x and y variables. From that, the domain of $f(x)$ which is $\{x:x\in\mathbb{R}\}$ becomes the range of $g(x)$, $\{y:y\in\mathbb{R}\}$ and the range of $f(x)$, $\{y:y>0\}$ becomes the domain of $g(x)$, $\{x:x>0\}$. Thus, the graph $f(x)$ is the mirror image of the graph of $g(x)$ on the line $y=x$ as shown in Figure 9.4.

Consider the exponential function $b^y=x$ where the base $b>0$. The exponent y is said to be the logarithm of x to base b and it is written as $y=\log_b x$. That is, $y=\log_b x$ is equivalent to $b^y=x$.

The logarithm of a number x to base 10 is written as $y=\log_{10} x$ and it is known as the common logarithm while the logarithm of a number x to base e written as $\log_e x$ or $\ln x$ is called the natural logarithm.

Properties of logarithmic functions

Using the fact that $b^1=b$

In logarithmic form is written as:

$$\log_b b=1$$

Also, $b^0=1$

In logarithmic form is:

$$\log_b 1=0$$

Now, let $\log_a x=m$ (9.1)

$$\log_a y=n \quad (9.2)$$

Write equation (9.1) and (9.2) in exponential form

$$x=a^m \quad (9.3)$$

$$y=a^n \quad (9.4)$$

Multiplying equation (9.3) by (9.4)

$$xy=a^m\times a^n$$

$$xy=a^{m+n} \quad (9.5)$$

In logarithmic form, equation (9.5) is written as $\log_a(xy)=m+n$

But $m=\log_a x$ and $n=\log_a y$

$$\log_a(xy)=\log_a x+\log_a y$$

Then divide equation (9.3) by (9.4) to get

$$\frac{x}{y}=\frac{a^m}{a^n}$$

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$$\frac{x}{y} = a^{m-n}$$

Introduce $\log_a$ both sides:

$$\log_a\left(\frac{x}{y}\right) = \log_a a^{m-n}$$

$$\log_a\left(\frac{x}{y}\right) = (m-n)\log_a a, \text{ but } \log_a a = 1$$

$$\log_a\left(\frac{x}{y}\right) = (m-n)$$

But $m = \log_a x$ and $n = \log_a y$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Using equation (9.3), $x = a^m$ so that

$\log_a x = m$

Then, multiplying to exponent n in equation $x = a^m$ throughout, to have:

$$x^n = \left(a^m\right)^n$$

$$x^n = a^{mn} \qquad (9.6)$$

Apply logarithm to base a both sides of the equation (9.6),

$\log_a x^n = mn$, but $m = \log_a x$

$\therefore \log_a x^n = n\log_a x$

Change of base in logarithms

Let $\log_a b = x$.

Then $b = a^x$

Taking logarithm to base c in both sides.

$$\log_c b = \log_c a^x$$

$$\log_c b = x\log_c a$$

$$x = \frac{\log_c b}{\log_c a}, \text{ but } x = \log_a b$$

$$\therefore \log_a b = \frac{\log_c b}{\log_c a}$$

This relationship can be used to evaluate the logarithm of a number to a base other than base 10 or base e.

From $\log_a b = \frac{\log_c b}{\log_c a}$

If $c = b$, then,

$$\therefore \log_a b = \frac{\log_c b}{\log_c a} = \frac{1}{\log_b a}$$

Example 9.7: Evaluate

a) $\log_8 2$ b) $\log_7 3$

Solution:

a) From $\log_8 2 = \frac{1}{\log_2 8}$

$$= \frac{1}{\log_2 2^3}$$

$$= \frac{1}{3\log_2 2}$$

$$= \frac{1}{3(1)} = \frac{1}{3}$$

$$\therefore \log_8 2 = \frac{1}{3}$$

b) From $\log_7 3 = \frac{\log_{10} 3}{\log_{10} 7}$

$$= 0.564575$$

or

$$\log_7 3 = \frac{\log_e 3}{\log_e 7} = \frac{\ln 3}{\ln 7}$$

$$= 0.564575$$

$$\therefore \log_7 3 = 0.564575$$

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Summary

Properties of logarithmic functions

a) $\log_b b = 1$

b) $\log_b a^x = x \log_b a$

c) $\log_b xy = \log_b x + \log_b y$

d) $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

e) $\log_b 1 = 0$

f) $\log_a b = \frac{\log_c b}{\log_c a}$

g) $\log_a b = \frac{1}{\log_b a}$

Properties of natural logarithmic functions

a) $\ln 1 = 0$

b) $\ln a^x = x \ln a$

c) $\ln(xy) = \ln x + \ln y$

d) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

e) $\ln e = 1$

Exercise 9.5

1. Rewrite each expression as a single logarithm

   a) $p \log_a x + \frac{1}{q} \log_a y$

   b) $\ln \frac{x}{\sqrt{x+2}} + \ln \frac{\sqrt{x^2-4}}{x^2}$

   c) $8 \log\left(x^2 - 1\right) - 4 \log(x+1)$

2. Simplify each expression

   a) $\frac{1}{3} \log_2 64$

   b) $\frac{\log 10^{2x^2-6x+4}}{2x-4}$

   c) $10 \log 2 - 2 \log 32$

3. Solve for x in the following equations:

   a) $7^{2x+1} = 3^{x-2}$

   b) $e^{1-3x} = 5^{2x}$

4. Show that $(\log_x y)(x \log_y z) = x \log_x z$

5. Given that

   $2 + \log_a b + 3 \log_a y = 2 \log_a\left(a^2 y\right)$,

   where $a \neq 0$, $b \neq 0$ $y \neq 0$, express y in terms of a and b without logarithms.

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The graphs of logarithmic functions

The graph of $y = \log_b x$ is the image of the graph of $y = b^x$ on the line $y = x$ which stands as a mirror. See Figure. 9.4

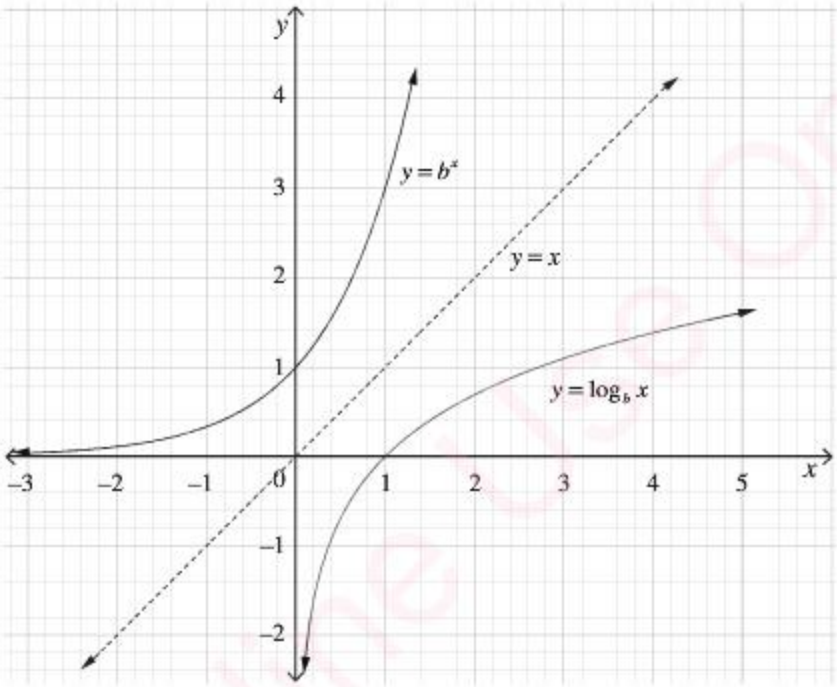

Figure 9.4: Graph of a logarithmic function as a reflection of exponential function

Example 9.8: Draw the graph of $f(x) = \log_2 x$

Solution:

Table 9.3 The table of values for the function $f(x) = \log_2 x$

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$f(x) = \log_2 x$	–4	–3	–2	–1	0	1	2	3	4

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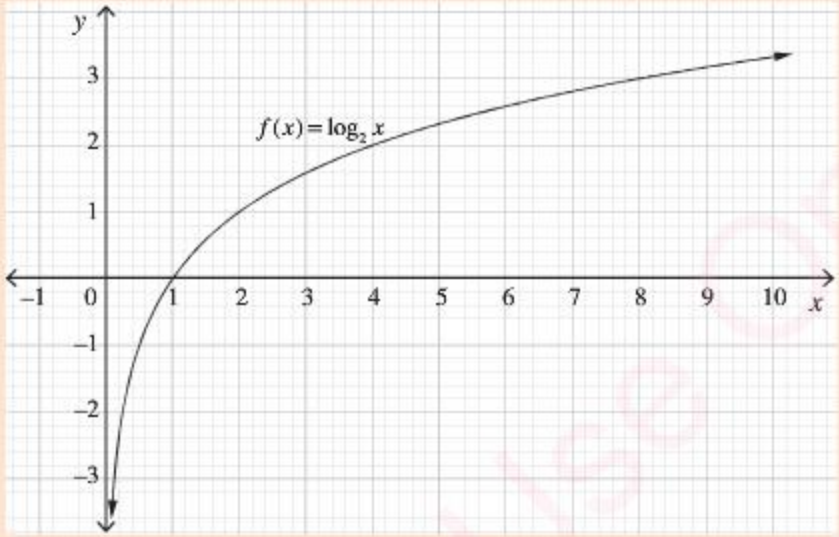

Figure 9.5 Graph of $f(x)=\log_2 x$

Domain and range of logarithmic functions

The domain of a logarithmic function $f(x)=\log_b g(x)$ is the set of all positive real numbers; i.e. Domain $\{x: g(x)>0\}$, while range of a logarithmic function $f(x)=\log_b g(x)$ is the set of all real numbers; that is Range $=\{y: y\in\mathbb{R}\}$.

Example 9.9: State the domain and range of the logarithmic function given in Example 9.8.

Solution

From the graph drawn in Figure. 9.5,

Domain: the graph in Figure 9.5 does not touch the y – axis and continue towards positive x – axis with no end, thus the Domain $=\{x: x>0\}$.

Range: the graph in Figure 9.5 continue on both positive and negative y – axis with no end, then the Range $=\{y: y\in\mathbb{R}\}$.

The behaviour of graphs of logarithmic functions can be explained by considering the graph of $y=\log x$. To sketch the graph of $y=\log x$ the following table of values is used.

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Table 9.4: Table of value for $y = \log x$

x	0.001	0.01	0.1	1	10	100	1000
y	–3	–2	–1	0	1	2	3

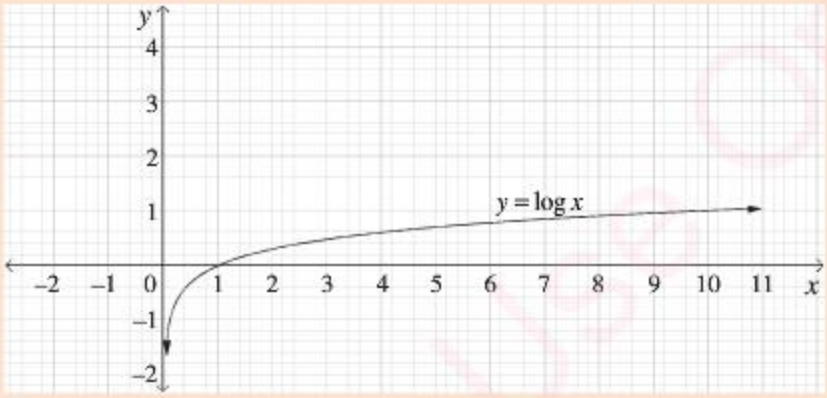

Figure. 9.6 The graph of $y = \log x$

From the graph (Figure. 9.6), it is observed that as $x \to 0$, $\log x \to -\infty$ which means the graph is asymptotic to y – axis (i.e. the line $x = 0$ is the vertical asymptote). The graph crosses at the point $(1, 0)$ which is called the x – intercept. The domain is the set of all positive real numbers that is $\{x : x > 0\}$ and the range is the set of all real numbers. When the value of $x \to 0$ from positive side the function decreases, while when the value of $x \to \infty$ the function increases.

The difference between common and natural logarithmic functions

The difference between common and natural logarithmic functions is found when changing its base. Common logarithmic function is a logarithmic function to base 10, written as $f(x) = \log_{10} x$ or $f(x) = \log x$. The symbol $\log x$ (with no subscript) is often used as a common form for $f(x) = \log_{10} x$, for $x > 0$.

Natural logarithmic function is a logarithmic function in the form $f(x) = \log_e x$, for $x > 0$ or $f(x) = \ln x$, for $x > 0$ whereby e is a Naperian constant whose numerical value is approximately 2.71828

Exercise 9.6

State domain and range of the following:

1. $f(x) = \log (x + 2)$
2. $f(x) = \ln(1 - x)$

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Draw the graph of the following:

3. $g(x)=\log_3 x$
4. $g(x)=\log(2+x)$
5. $g(x)=\ln x$

9.4 Differentiation of logarithmic and exponential functions

a) *Derivatives of natural logarithmic functions*

Given $y=\ln x$, in exponential form, it can be written as $e^y=x$

Differentiating with respect to x yields:

$$\frac{d}{dx}(e^y)=\frac{d}{dx}(x)$$

$$e^y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{e^y}, \text{ but } e^y=x$$

$$\frac{dy}{dx}=\frac{1}{x}$$

Generally, when differentiating the natural logarithmic function in the form of

a) $y=\ln ax$

b) $y=\ln(ax+b)$

The chain rule of differentiation will be more preferably to be applied as follows:

a) Given $y=\ln ax$, where $ax>0$,

let $u=ax \Rightarrow \frac{du}{dx}=a$ and

$$y=\ln u \Rightarrow \frac{dy}{du}=\frac{1}{u}.$$

Then, $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$$=\frac{1}{u}\times a=\frac{a}{u},$$

but $u=ax$ then,

$$\frac{dy}{dx}=\frac{a}{ax}=\frac{1}{x}.$$

b) From, $y=\ln(ax+b)$, where $(ax+b)>0$.

Let $u=ax+b$

$$\frac{du}{dx}=a$$

Now, $y=\ln u$

$$\Rightarrow \frac{dy}{du}=\frac{1}{u}$$

By chain rule;

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

This leads to $\frac{dy}{du}=a\times\frac{1}{u}$.

But $u=ax+b$.

Then $\frac{dy}{dx}=\frac{a}{ax+b}$

Generally $\frac{d}{dx}\ln[f(x)]=\frac{f'(x)}{f(x)}$

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Example 9.10: Find the first derivative of the following:

a) $f(x)=\ln(3x+1)$

b) $g(x)=\ln(2x-3)$

Solution

a) Given $y=f(x)=\ln(3x+1)$

Let $u=3x+1$

$$\frac{du}{dx}=3$$

It follows that

$$y=\ln u \quad \text{and} \quad \frac{dy}{du}=\frac{1}{u}$$

Using chain rule,

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$\frac{dy}{dx}=3\times\frac{1}{u}$$

$$=3\times\frac{1}{3x+1}$$

$$\therefore \frac{dy}{dx}=\frac{3}{3x+1}$$

b) Given $y=g(x)=\ln(2x-3)$

Let $u=2x-3$

$$\Rightarrow \frac{du}{dx}=2$$

It follows that

$$y=\ln u \text{ and } \frac{dy}{du}=\frac{1}{u}$$

Using chain rule,

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$\frac{dy}{dx}=2\times\frac{1}{u}$$

$$=2\times\frac{1}{2x-3}$$

$$\therefore \frac{dy}{dx}=\frac{2}{2x-3}$$

b) ***Derivatives of common logarithmic functions***

Steps:

i. Convert common logarithm into exponential form

ii. Convert exponential into logarithm form

iii. Differentiate the natural logarithm obtained

Consider the logarithmic function $y=\log_a x$, then the function in exponential form becomes:

$$a^y=x.$$

Converting this into natural logarithm yields:

$$\ln a^y=\ln x \Rightarrow y\ln a=\ln x$$

Differentiating implicitly with respect to x gives:

$$\frac{d}{dx}(y\ln a)=\frac{d}{dx}(\ln x)$$

$$\ln a\frac{dy}{dx}=\frac{1}{x}\Rightarrow\frac{dy}{dx}=\frac{1}{x\ln a}.$$

Thus $\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}$

Alternatively, by conversion from base a to base e

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$$y=\frac{\log_e x}{\log_e a}=\frac{\ln x}{\ln a}, \text{ then}$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$$

$$\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}$$

Example 9.11: Differentiate the following with respect to x:

a) $f(x)=\log_{10} x$

b) $g(x)=\log_{10} e^{2x+3}$

c) $h(x)=\log_{10}(2x^2+4x+5)$

Solution

a) From $y=f(x)=\log_{10} x$

$\Rightarrow x=10^y$

Apply natural logarithm throughout

$\ln x=\ln 10^y$

$\ln x=y\ln 10$

Differentiate with respect to x both sides

$$\frac{1}{x}=(\ln 10)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{1}{x\ln 10}$$

b) From $y=g(x)=\log_{10} e^{2x+3}$

$\Rightarrow 10^y=e^{2x+3}$

Apply natural logarithm throughout

$y\ln 10=2x+3$

Differentiate both sides with respect to x

$$\ln 10\frac{dy}{dx}=2$$

$$\therefore \frac{dy}{dx}=\frac{2}{\ln 10}$$

c) $y=h(x)=\log_{10}(2x^2+4x+5)$

$10^y=2x^2+4x+5$

Apply natural logarithm throughout

$y\ln 10=\ln(2x^2+4x+5)$

Differentiate both sides with respect to x

$$\ln 10\frac{dy}{dx}=\frac{4x+4}{2x^2+4x+5}$$

$$\therefore \frac{dy}{dx}=\frac{4x+4}{(2x^2+4x+5)\ln 10}$$

c) *Derivatives of exponential functions*

When differentiating the exponential function of the form $f(x)=a^x$ where $a>0$, the concept of natural logarithmic functions may be used as follows:

Let $y=a^x$.

Introduce natural logarithm on both sides

$$\ln y=\ln a^x \Rightarrow \ln y=x\ln a$$

Now, $\frac{d}{dx}(\ln y)=\frac{d}{dx}(x\ln a)$

$$\frac{1}{y}\frac{dy}{dx}=\ln a$$

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$$\frac{dy}{dx} = y \ln a, \text{ but } y = a^x$$

$$\frac{dy}{dx} = a^x \ln a$$

$$\therefore \frac{d}{dx}(a^x) = a^x \ln a$$

If $a = e$, then $\frac{d}{dx}(e^x) = e^x \ln e$

But $\ln e = 1$

$$\therefore \frac{d}{dx}(e^x) = e^x$$

Example 9.12: Find $\frac{dy}{dx}$, if $y = 2^x$.

Solution

Given $y = 2^x$

Apply natural logarithm on both sides

$$\ln y = \ln 2^x$$

$$\ln y = x \ln 2$$

Differentiate both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \ln 2$$

$$\frac{dy}{dx} = y \ln 2.$$

But $y = 2^x$

$$\therefore \frac{dy}{dx} = 2^x \ln 2$$

Example 9.13: Find derivative of

$f(x) = 10^{2x}$

Solution

Let $y = 10^{2x}$

Apply natural logarithm both sides to get

$$\ln y = \ln 10^{2x}$$

$$\ln y = 2x \ln 10$$

Differentiate both sides with respect to x

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(2x \ln 10)$$

$$\frac{1}{y}\frac{dy}{dx} = 2 \ln 10$$

$$\frac{dy}{dx} = 2 \ln 10 \times y$$

But $y = 10^{2x}$

$$\Rightarrow \frac{dy}{dx} = 2 \ln 10 \times 10^{2x}$$

$$\therefore \frac{dy}{dx} = 10^{2x} \ln 100$$

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Exercise 9.7

1. Find the first derivative of the following:
   a) $f(x)=\ln(9x+7)$
   b) $g(x)=\ln\left(\frac{1}{x+2}\right)$
   c) $h(x)=\ln\left(\frac{x+1}{x+2}\right)$
2. Find the first derivative of the following:
   a) $f(x)=\log_{10}(x+1)$
   b) $g(x)=\log_{10}e^{4x+7}$
   c) $h(x)=\log_{10}(x^2+2)$
3. Find $\frac{dy}{dx}$ for the following functions:
   a) $y=3^x$
   b) $y=5^{2x}$
   c) $y=100^{-x}$
4. Find the derivative of the following:
   a) $f(x)=x^2\ln 2x$
   b) $h(x)=2x^3(2^x)$
   c) $g(x)=\frac{3^x}{\log x}$
5. Find the first and second derivatives of the following:
   a) $f(x)=3^{-x}$
   b) $g(x)=\log(1-x)$
   c) $h(x)=\ln(2x+4)$

9.5 Integration of the form $f(x)=\frac{1}{ax+b}$

Before integrating the function of the form $f(x)=\frac{1}{ax+b}$, first consider the integration of the form $f(x)=\frac{1}{x}$.

Recall that if $y=\ln x$, then $\frac{dy}{dx}=\frac{1}{x}$.

This implies that $\frac{d}{dx}(\ln x)=\frac{1}{x}$

Integrate both sides with respect to x

$$\int\frac{d}{dx}(\ln x)dx=\int\frac{1}{x}dx$$

$$\ln|x|+c=\int\frac{1}{x}dx$$

$$\therefore\int\frac{1}{x}dx=\ln|x|+c$$

Now, consider $\int\frac{dx}{ax+b}$

Substituting

$u=ax+b$ and $\frac{du}{dx}=a\Rightarrow dx=\frac{du}{a}$

It follows that

$$\int\frac{dx}{ax+b}=\frac{1}{a}\ln|u|+c, \text{ but } u=ax+b$$

$$\therefore\int\frac{dx}{ax+b}=\frac{1}{a}\ln|ax+b|+c$$

where $(ax+b)>0$

Generally,

$$\int\frac{dx}{ax+b}=\frac{\ln|ax+b|}{\frac{d}{dx}(ax+b)}+c=\frac{1}{a}\ln|ax+b|+c$$

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Example 9.14: Find the following:

a) $\int \frac{dx}{2x+3}$

b) $\int \frac{dx}{x+2}$

Solution

a) $\int \frac{dx}{2x+3}$

Let $u = 2x+3$

$\frac{du}{dx} = 2$

$dx = \frac{du}{2}$

$\int \frac{dx}{2x+3} = \int \left(\frac{du}{2} \div u\right)$

$= \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2}\ln|u| + c$

But $u = 2x+3$

$\therefore \int \frac{dx}{2x+3} = \frac{1}{2}\ln|2x+3| + c$

Altenatively,

$\int \frac{dx}{2x+3} = \frac{\ln|2x+3|}{\frac{d}{dx}(2x+3)} + c$

$= \frac{1}{2}\ln|2x+3| + c$

$\therefore \int \frac{dx}{2x+3} = \frac{1}{2}\ln|2x+3| + c$

b) $\int \frac{dx}{x+2}$

Let $u = x+2$

$\frac{du}{dx} = 1$

$du = dx$

Now,

$\int \frac{dx}{x+2} = \int \frac{du}{u}$

$= \ln|u| + c$

but $u = x+2$

$\therefore \int \frac{dx}{x+2} = \ln|x+2| + c$

Altenatively,

$\int \frac{dx}{x+2} = \frac{\ln|x+2|}{\frac{d}{dx}(x+2)} + c$

$= \ln|x+2| + c$

$\therefore \int \frac{dx}{x+2} = \ln|x+2| + c$

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Exercise 9.8

1. Integrate the following with respect to x:

   a) $f(x)=\dfrac{1}{5x-1}$

   b) $f(x)=\dfrac{1}{10x-3}$

   c) $g(x)=\dfrac{1}{3x+7}$

   d) $h(x)=\dfrac{1}{5x+6}$

2. Evaluate the following:

   a) $\int_{-1}^{1}\dfrac{1}{2x-6}dx$

   b) $\int_{3}^{4}\dfrac{1}{7-15x}dx$

9.6 Applications of Exponential and Logarithmic Functions

Exponential and logarithmic functions can be used to analyse various real life problems such as compound interest, depreciation, population growth and decay, radioactive decay and cooling.

Compound interests

The rate at which compound interest accumulates depends on the frequency of compounding. The higher the number of compounding periods, the greater the compound interest.

Let P represents the principal, t be the number of years, r % the rate of interest, A_n be the accumulated amount in a given number of years. The formula for calculating compound interest from section 3.2 is: $A_n=P\left(1+\dfrac{r}{100n}\right)^{nt}$, where n is the number of times the interest is compounded per year.

Example 9.15: How many years will it take for Mary to have the amount of money accumulated twice the principal in a bank that provides an interest of 10% compounded annually?

Solution

Given,

$r=10$, $P=$ Principal, $A=2P$,

$n=1$ and $t=$ number of years

$$A_n=P\left(1+\frac{r}{100n}\right)^{nt}$$

$$2P=P\left(1+\frac{10}{100}\right)^{t}$$

$$2=(1.1)^{t}$$

Applying common logarithm to have,

$$\log 2=\log(1.1)^{t}$$

$$\log 2=t\log(1.1)$$

$$t=\frac{\log 2}{\log 1.1}$$

$$t=7.273$$

Therefore, it will take 7 years and 4 months for Mary to receive accumulated amount twice its principal.

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Example 9.16: Kazimoto deposited Tsh 5250 at a bank offering 8% interest compounded semi-annually. Find the amount accumulated and the compound interest after 3 years.

Solution

Using $A_n = P\left(1+\frac{r}{100n}\right)^{nt}$

where,

$$P = 5250, \quad r = 8, \quad n = 2, \quad t = 3$$

Now, $A = 5250\left(1+\frac{8}{100\times 2}\right)^{2\times 3}$

$$= 5250\left(1.04\right)^{6}$$

∴ Amount accumulated is Tsh 6642.90

Now, for compound interest (I):

$$I = A - P$$

$$I = 6642.90 - 5250$$

∴ Compound interest, I is Tsh 1392.90

Note that as the interest gets compounded continuously, the amount A after t years will be an exponential function with a continuous compounding interest growing exponentially.

Depreciation

Assets depreciate their values with time. The decrease in assets' values depends on a number of factors and unfavourable market conditions. However, not all assets may depreciate with time, land is an example. If an asset depreciates at the rate R per annum, then from the initial value P, its value in one year will be $P(1-R)$ where $R = r\%$. For each successive year, the value is multiplied by the same factor as follows:

Year 1 : $A = P(1-R)$

Year 2 : $A = P(1-R)^2$

Year 3 : $A = P(1-R)^3$

Therefore, after t years, its amount will be

$A = P(1-R)^t$

Example 9.17: In 2015, a customer bought a mobile phone for Tsh 400,000. The phone loses its value at the rate of 10% and 25% for the first and second year, respectively. Calculate the new price of the phone.

Solution

Given: $P = \text{Tsh } 400{,}000$

$$R_1 = 10\%, \; R_2 = 25\%$$

Then, from $A = P(1-R)^t$

After one year its price will be

$A = 400000(1-0.1) = \text{Tsh } 360{,}000$

After two years its amount will be

$A = 360000(1-0.25) = \text{Tsh } 270{,}000.$

Exercise 9.9

1. Hamisi bought a computer for Tsh. 125,000, if the computer depreciates at 25 % each year. What is the value of computer after 2 years?
2. A Sony radio whose cost is Tsh 220,000 depreciated to Tsh 12,000 in 5 years. Find the depreciation rate.
3. Janeth bought a car at Tsh 6,000,000, if the car depreciates at the rate of 20 % per year. How long will it take for the value of the car to be Tsh 3,000,000?

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4. After 25 years the motorcycle bought by Tumaini VICOBA was depreciated to Tsh 35,894.90 at the rate of 10 %. What was the original price of the motorcycle?
5. How long will it take for the price of the calculator to be half of the original price if the depreciation rate is 6 % per year?
6 Mwananchi SACCOS deposited Tsh 1,250,000 in an account that pays 9 % compounded monthly. How much will be in their account at the end of 2 years?
7 Flaviana invested Tsh 60,000 at 9 % compounded quarterly for 3 years.
   a) How much interest did she earn at the end of 3 years?
   b) How much interest did she earn in the second year?

Real life problems related to exponential functions

Population growth and decay

Population growth and/or decay is a good example of exponential function. Suppose $P(t)$ is the size of population at any time t. It can be described exponentially when the rate of change is proportional to an increase or decrease of the population size.

That is:

$\frac{dP}{dt} \alpha \pm P \Rightarrow \frac{dP}{dt} = \pm kP$, where k is proportionality constant.

From $\frac{dP}{P} = \pm k dt$ then, through separation of variables, we obtain:

$$\int \frac{dP}{P} = \pm k \int dt \Rightarrow \ln|P| = \pm kt + c$$

$P(t) = ce^{\pm kt}$, at $t = 0$, we obtain

$P(t) = P_0 e^{\pm kt}$.

Therefore, the exponential function model is given by $P(t) = P_0 e^{kt}$ for population growth and $P(t) = P_0 e^{-kt}$ for population decay, where P_0 is an initial population, k is a constant, and t is the time.

Example 9.18: The population growth in town C is given in Table 9.5. Find the population in the year 2000.

Table 9.5: The population growth in town C

Year	Population
1985	20,000,000
1995	24,000,000
2000	-

Solution

From

$$P(t) = P_0 e^{kt}$$

$t = 0, \quad P_0 = 20,000,000$

$t = 10, \ P(t) = 24,000,000$

So that, $24,000,000 = 20,000,000 \times e^{10k}$

$$k = \frac{\ln 1.2}{10} = 0.0182$$

When $t = 15$ years,

$$P(15) = 20,000,000 \times e^{0.0182 \times 15}$$

$$= 26,278,005$$

Therefore, the population in town C in the year 2000 was 26,278,005.

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Example 9.19: The population of a certaion region decreases according to the relationship $P(t) = P_0 e^{-0.04t}$. If the initial population was 800, 000, how long will it take for the population to decline to three quarter of its initial number?

Solution

Given, $P_0 = 800\,000$,

$$P(t) = \frac{3}{4}P_0 = 600\,000,\ t = ?$$

From $P(t) = P_0 e^{-0.04t}$

$$600\,000 = 800\,000 \times e^{-0.04t}$$

$$t = \frac{\ln 0.75}{-0.04} = 7.19 \text{ years}$$

After 7.19 years, the population of ministry officers in Dar es Salaam will be three quarter of its original population.

Exercise 9.10

1. As of 1995 demographic survey, the population was expected to double within 50 years, calculate the constant of proportionality (k).
2. The population of Iringa region at the end of the year 2000 was 500,000. The population increases at the rate of 20 % per year.
   a) Write down the population equation, $P(t)$ of Iringa region after the year 2000.
   b) What will be the population at the end of the year 2050?
3. The current population of lizards in a certain island is 1,000. If they breed at a constant rate of 10 % per year, what will be the population of the lizards in the Island after 7 years? Give the answer in four significant figures.

Radioactive decay

In nature, there is a large number of atomic nuclei that can spontaneously emit elementary particles or nuclear fragments. Such a phenomenon is called radioactive decay.

The law of radioactive decay is given by the equation:

$$\frac{dN}{dt} = -kN,$$

where N is the amount of radioactive material and k is a positive constant depending on radioactive substance. The minus sign on the right side means that the amount of radioactive material $N(t)$ decreases exponentially with time.

$$\frac{dN}{N} = -kt \Rightarrow \int \frac{dN}{N} = -k\int dt$$

$$\ln|N| = -kt + c$$

$$N = e^{-kt+c}$$

$$N(t) = e^{-kt}e^{c} \quad \text{at } t = 0,\ e^{c} = N_o$$

The radioactive decay equation is written as:

$$N(t) = N_0 e^{-kt}$$

Furthermore, two parameters are introduced in the given equation. The half-life or half-life period T of radioactive material is the time required for a material to decay to one-half of the initial value.

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Hence,

$$N(T)=\frac{N_0}{2}=N_0e^{-kT}$$

$$\Rightarrow \frac{N_0}{2}=N_0e^{-kT}$$

$$\frac{1}{2}=e^{-kT}$$

Apply natural logarithm both sides to get,

$$\ln\left(\frac{1}{2}\right)=-kT$$

$$-\ln 2=-kT$$

Thus, the half life is given by:

$$\therefore T=\frac{\ln 2}{k}$$

Example 9.20: The initial mass of an iodine isotope was 200 g. Determine the mass of iodine after 30 days if the half life of the isotope is 8 days.

Solution

From the radioactive law:

$N(t)=N_0e^{-kt}$, but $k=\frac{\ln 2}{T}$ where

$T=8$ days, $k=\frac{\ln 2}{8}=0.0866$

$N(30)=200e^{-0.0866\times 30}=14.88\,g$.

Therefore, the mass of iodine isotope in 30 days is 14.9 g.

Exercise 9.11

1. Suppose a Bismuth 210 sample originally has a mass of 800mg with a half life of 5 days.
   a) Find a formula for mass remaining after t days.
   b) Find the remaining mass after 30 days
   c) When is the mass reduced to 1 mg?
2. Pre-historic cave paintings were discovered at Amboni caves in Tanga. The paint contained 12 % of the original carbon 14. Estimate the age of painting in years given that the constant $k=0.000121$.
3. The radioactive isotope Indium-111 is often used for diagnosis and imagining in nuclear medicine. Its half life is 2.8 days. What was the initial mass before the decay, if the mass in two weeks was 5 grams?
4. Potassium 42 is a radioactive element that is often used as a tracer in biological experiment. Its half life is approximately 12.5 hours.
   a) If N_0 milligram of potassium 42 are initially present, write a formula for the number of milligrams N present after t hours
   b) What percentage decrease in the amount of potassium 42 occurs in a sample during 1 hour?
   c) How much of 1 milligram sample of potassium 42, is left after 5 hours?
5. The half life of radium is 1656 years.
   a) If N_0 grams of radium are initially present, write the formula for the number of grams N present after t years.

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b) What percent decrease occurs in the amount of radium in a sample over 1 year?

c) How much of a 1-gram sample of radium is present after 20 years?

Newton's law of cooling

The Newton's law of cooling states that the rate at which the temperature of an object falls is proportional to the amount by which its temperature exceeds that of its surroundings.

The Newton's law of cooling is an example of exponential function. Suppose T_t is the temperature of an object at any time t, T_s is the temperature of the surrounding. It can be described exponentially when the rate of change is proportional to the excess temperature between the object and the surrounding.

Mathematically,

$$\frac{dT}{dt}\alpha\,(T_t - T_s) \Rightarrow \frac{dT}{dt} = -k(T_t - T_s)$$

where k is the proportionality constant.

$\frac{dT}{T_t - T_s} = -kdt$ then apply integration on both sides to get

$$\int \frac{dT}{T_t - T_s} = -k\int dt$$

$$\ln|T_t - T_s| = -kt + c$$

$T_t - T_s = e^{-kt+c}$ at $t = 0, T_t = T_0$ then,

$$T_0 - T_s = e^{-k(0)+c}$$

$$T_0 - T_s = e^{c}$$

$$T_t = T_s + (T_0 - T_s)e^{-kt}$$

Therefore, the exponential function model of Newton's law of cooling is given by $T_t = T_s + (T_0 - T_s)e^{-kt}$, where T_0 is the initial temperature of an object.

Example 9.21: Suppose the temperature of an object falls from $240\,^{\circ}C$ to $180\,^{\circ}C$ in 50 minutes, in a surrounding temperature of $20\,^{\circ}C$. Calculate the time it takes to reach $100\,^{\circ}C$ correct to three significant figures.

Solution

Using $T_t = T_s + (T_0 - T_s)e^{-kt}$ where $t = 0$ and $T_0 = 240°C$, and when $t = 50$, $T_1 = 180°C$ and $T_s = 20°C$,

substituting into equation above gives;

$$180 = 20 + (240 - 20)e^{-k(50)}$$

$$160 = 220e^{-50k}$$

$$\frac{160}{220} = e^{-50k}$$

Apply natural logarithm to both sides yealds;

$$\ln\left(\frac{16}{22}\right) = -50k$$

$$k = \frac{1}{-50}\ln\left(\frac{16}{22}\right) = -0.006369.$$

When $T_1 = 100\,^{\circ}C$, then

$$100 = 20 + (240 - 20)e^{-0.006369t}$$

$$80 = 220e^{-0.006369t}$$

$$\frac{80}{220} = e^{-0.006369t}$$

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Apply natural logarithm both sides

$$\ln\left(\frac{8}{22}\right) = -0.006369t$$

$$-1.011600912 = -0.006369t$$

$$t = \frac{1.011600912}{0.006369}$$

$t = 158.83$ minutes (or 158 minutes and 50 seconds).

Therefore, it will take 158 minutes and 50 seconds to reach $100\,^{\circ}C$.

Example 9.22: A rod of iron is heated in a forge to a temperature of $1080\,^{\circ}C$. The rod is then plunged in a bucket of chilled water with a temperature of $25\,^{\circ}C$. After 25 seconds, the temperature of the iron rod drops to $82.5\,^{\circ}C$. Find the cooling rate for this iron rod in water.

Solution

From $T_t = T_s + (T_0 - T_s)e^{-kt}$, by making e^{-kt} the subject of the equation then,

$$e^{-kt} = \frac{T_t - T_s}{T_0 - T_s}$$ apply natural logarithm to both sides

$$k = \frac{1}{-t}\ln\left|\frac{T_t - T_s}{T_0 - T_s}\right|$$ where $t = 25$ sec, $T_t = 82.5\,^{0}C, T_s = 25\,^{0}C, T_0 = 1080\,^{0}C$

$$k = \frac{1}{-25}\ln\left|\frac{82.5-25}{1080-25}\right| = \frac{1}{-25}\ln(0.0545) = 0.1164$$

Therefore, the cooling rate for iron rod in water is 0.1164 per second.

Exercise 9.12

1. A soup pot starts at a temperature of $385K$ and the surrounding temperature is $293K$. If the cooling constant is $k = 0.00148$ per second, what will be the temperature of the soup pot after 35 minutes?
2. The temperature of an object falls from $175\,^{\circ}C$ to $95\,^{\circ}C$ in 45 minutes after being placed in a certain room temperature of $20°C$. After how many minutes will the temperature of the object fall to $40°C$?
3. The temperature of the body falls from $371K$ to $345K$ in 52 minutes after being placed in a room temperature of $298K$. Find the temperature of the body when the time is one hour and thirty minutes.
4. Suppose the temperature of an object falls from $220\,^{0}C$ to $110\,^{0}C$ in 52 minutes, in a surrounding temperature of $20°C$.

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a) Show that after t minutes, the temperature, T of an objects is given by $T = 20 + 200e^{-kt}$, where $k = \frac{1}{52}\ln\left(\frac{20}{9}\right)$.

b) Calculate the time it takes to reach the temperature of $80\ ^{o}C$.

5. A rod of silver is heated in a forge to a temperature of $860K$. It is then plunged in a bucket of chilled water with a temperature of $282K$. After 24 seconds the temperature of the silver rod drops to $372.5K$. Find the rate of cooling for this silver rod in water.

Revision exercise 9

1. Sketch the graphs of the following and state the domain and range.

a) $f(x) = \frac{1}{2^x}$ b) $g(x) = \left(\frac{2}{3}\right)^x$

c) $h(x) = 5^x$ d) $f(x) = -\left(\frac{2}{3}\right)^x$

2. Solve for the value of x in the following equations:

a) $\log_3(x+1) + \log_3(x+3) = 1$ b) $\log_2(x+7) - \log_2 x = 3$

c) $\log_2 x^2 = (\log_2 x)^2$

3. Find the derivative of the following:

a) $f(x) = x^2e^{2x}$ b) $h(x) = e^{\sqrt{x}}$ c) $g(x) = e^{3x^2}$

d) $f(x) = e^{-4x+2}$ e) $g(x) = xe^{3x}$

4. Evaluate the following.

a) $\int_0^1 e^{-2x}dx$ b) $\int_{0.5}^1 e^{-3x+2}dx$ c) $\int_{-2}^{3} \frac{2}{1+2x}dx$

d) $\int_2^6 \frac{1}{x+3}dx$

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5. Integrate the following functions with respect to x:

   a) $f(x)=\dfrac{6}{5x+1}$ b) $h(x)=\dfrac{1}{x+1}$ c) $g(x)=\dfrac{3}{2x+3}$

6. John invested Tsh 50,000 at a bank offering 10 % per annum and it is compounded quarterly. Find the amount of the investment at the end of 5 years.

7. Suppose a teacher invests Tsh 200,000 in a savings account. The average interest rate is 4 % per annum and is compounded half-yearly. How much money will be in the account after 4 years?

8. A certain strain of bacteria that is growing on a kitchen counter doubles every 5 minutes. If initially the kitchen had one bacterium, how many bacteria could be present at the end of 96 minutes?

9. A scientist started with 100 bacteria in an experiment. After 5 days, the scientist discovered that the population had grown to 350 bacteria. Determine an equation for this bacteria population.

10. How much money will be accumulated if Tsh 100,000 is deposited in the account that earns interest of the rate of 7.5 % annually for 54 months, if :
    a) interest is compounded quarterly
    b) interest is compounded monthly

11. The growth of a certain bacteria is given by the equation $P=P_0e^{0.195t}$. If initially there are 500 bacteria present and t is given in hours,
    a) how many bacteria are there after two days?
    b) how long will it take to have 10,000 bacteria?

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Chapter
Ten

Matrices

Introduction

Numbers, symbols, or expressions can be presented into rows and columns to simplify understanding and interpretation of the quantities they represent. In mathematics, this form of arrangement of numbers, symbols or expressions is referred to as Matrices. The application of matrices is common in economics, engineering and demography, among many others. In this chapter, you will learn the meaning, types and properties of a matrix, operations with matrices, determinant of a matrix, inverse of a matrix, matrix solution of systems of linear equations and real life problems involving matrices.

10.1 Introduction to matrices

Matrix refers to the rectangular arrays of numbers/symbols/objects which are arranged in rows and columns. The plural of the word matrix is matrices. It helps to identify corresponding values easily.

$$A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{matrix}\leftarrow row_1\\ \leftarrow row_2\\ \leftarrow row_3\end{matrix}$$

Order of matrices

The order of matrices is described by using number of rows and columns. The number of rows is normally denoted by m and the number of columns is normally denoted by n. Now, the order of matrix is written as $m\times n$.

$$A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$\uparrow Column_1$ $\uparrow Column_2$ $\uparrow Column_3$

Matrices are usually denoted by capital letters such as A, B, C or any other letters. The numbers or letters which are enclosed in the brackets of a matrix are called elements of a matrix. The brackets used in matrices are parentheses $(\ )$ or a pair of square brackets $[\ ]$. For example, matrix A below has three rows and three columns:

For example, an element "e" in matrix A belongs to the second row and second column. The elements in the above data are said to form a rectangular array. In such array, elements across the matrix are called rows and elements downwards the matrices are called columns. Thus, matrix A is called "three by three" matrix written as "3×3".

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Different information may be arranged using order of matrices. For example, consider the number of students from form I to form IV taking Mathematics, Chemistry and Physics as shown in the Table 10.1.

Table 10.1 Number of students taking Mathematics, Chemistry and Physics

Class	Mathematics	Chemistry	Physics
Form I	90	90	90
Form II	60	60	60
Form III	85	85	85
Form IV	58	58	58

From Table 10.1, the given information can be arranged in the form of matrix as described in matrix C.

$$C=\begin{pmatrix}90 & 90 & 90\\ 60 & 60 & 60\\ 85 & 85 & 85\\ 58 & 58 & 58\end{pmatrix}$$

In matrix C the horizontal array shows the order of rows and the vertical array shows the columns of the matrix. Therefore, matrix C is of order 4×3. Generally, the matrix concepts are very useful in summarizing data.

Activity 10.1

An electronic company produces transistors, resistors and the computer chips. Each transistor requires 4 units of Aluminium, 2 units of Bronze and 2 units of Zinc. Each resistor requires 1,2 and 3 units of Aluminium, Bronze and Zinc, respectively. Each computer chip requires 6, 2, and 4 units of Aluminium, Bronze and Zinc, respectively. Write this information in matrix form.

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Types of matrices

a) Row matrix

A type of matrix with only one row. For example, $A = [1\ 2\ 3]$ has three columns and one row and is 1×3 matrix.

b) Column matrix

A type of matrix with only one column. For example, the matrix defined by $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has three rows and one column, which is 3×1 matrix.

c) Sqar e matrix

This is a matrix with the same number of columns and rows. For example, matrix $B = \begin{pmatrix} 1 & 2 & 7 \\ 2 & 2 & 3 \\ 3 & 5 & 7 \end{pmatrix}$ is a square matrix since the number of rows and the number of columns are equal; i.e. there are 3 rows and 3 columns. It is a 3×3 matrix.

d) Identity matrix

This is a square matrix whose all elements in the leading diagonal are one and the rest of the elements are zero.

For example, matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

is an identity matrix since the leading diagonal elements are units while non-leading diagonal elements (the rest of elements) are all zero. It is therefore called 3×3 identity matrix. An identity matrix is denoted by the symbol I.

e) Zero or null matrix

A matrix whose all elements are zero is called a zero or null matrix.

For example, matrix $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

is called 'zero matrix' because all elements are zero.

f) Diagonal matrix

A square matrix is called a diagonal matrix if all the elements except those in the leading diagonal are zero. For example, matrices

$A = \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

are diagonal matrices of order 2×2 and 3×3, respectively.

g) Scalar matrix

This is a square matrix whose elements in the leading diagonal are equal or the same, and the rest of the elements are zero. For example,

$F = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ and $G = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix}$

are scalar matrices of the order 2×2 and 3×3, respectively.

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Equality of matrices

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be equal if they are of the same order and if all the corresponding entries are equal, mathematically written as $\boldsymbol{A} = \boldsymbol{B}$.

Example 10.1: Given the matrices $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & y & 9 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & x \\ 7 & 8 & 9 \end{pmatrix}$, find x and y values if $\boldsymbol{A} = \boldsymbol{B}$.

Solution

Since the two matrices are equal, then $x = 6$ and $y = 8$

Example 10.2: State the order and name the position of the elements represented by letters in the following matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 1 & r \\ s & 1 & 1 & 0 \\ 0 & t & 3 & 0 \end{pmatrix}$$

Solution

The order of the matrix is 3 by 4 and written as 3×4

The position of element r is in the first row and fourth column

The position of element s is in the second row and first column

The position of element t is in the third row and second column

Example 10.3: If $\begin{pmatrix} 2x-4 & 12 & 4c-7 \\ 3y+8 & 2b-3 & -14 \\ -10 & k+5 & 2a \end{pmatrix} = \begin{pmatrix} x-10 & 7z-2 & c+2 \\ y-4 & 3b+2 & -14 \\ -10 & 2k-1 & 10 \end{pmatrix}$, find the value of a, b, c, x, y and z.

Solution

Since the two matrices are equal, then their corresponding elements are equal.

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Therefore,

$2x-4=x-10$	$3x+8=y-4$	$7z-2=12$
$2x-x=-10+4$	$3y-y=-4-8$	$7z=14$
$\therefore\ x=-6$	$2y=-12$	$\therefore\ z=2$
	$\therefore\ y=-6$	
$2a=10$	$2b-3=3b+2$	$4c-7=c+2$
$\therefore a=5$	$2b-3b=2+3$	$4c-c=2+7$
	$\therefore\ b=-5$	$3c=9$
		$\therefore\ c=3$

Hence $a=5$, $b=-5$, $c=3$, $x=-6$, $y=-6$ and $z=2$.

Exercise 10.1

1. Ally, Neema and Sauda are the students in form six class. Ally has 8 counter books, 6 textbooks and 2 calculators. Neema has 7 counter books, 9 textbooks and 1 calculator. Sauda has 10 counter books, 5 textbooks and 3 calculators.
   a) Express this information in the matrix form
   b) State the order of the matrix in a) above

2. In the matrix $\boldsymbol{F}=\begin{pmatrix}8 & 6\\ 4 & 2\\ 7 & 1\end{pmatrix}$, find:
   a) the order of matrix $\boldsymbol{F}$
   b) the number of elements in matrix $\boldsymbol{F}$

3. If $\begin{bmatrix}2x & 2x-y\\ 3w & 2y+3z\end{bmatrix}=\begin{bmatrix}4 & 12\\ 15 & 8\end{bmatrix}$, find the values of x, y, w and z.

4. Construct a 4×4 identity matrix and a 3×3 scalar matrix whose scalar is 14.

5. State the type and order of the following matrices.
   a) $\begin{bmatrix}8\\4\\2\end{bmatrix}$ b) $\begin{pmatrix}7 & 9 & -4\end{pmatrix}$ c) $\begin{pmatrix}6 & 8 & -2\\ 2 & 9 & -3\end{pmatrix}$
   d) $\begin{pmatrix}2 & 4\\ 8 & 9\end{pmatrix}$ e) $\begin{pmatrix}12 & 0\\ 0 & 12\end{pmatrix}$ f) $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 3 & 9\end{pmatrix}$

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10.2 Operations with matrices

Addition and subtraction

Addition and subtraction of matrices can be done only for matrices of the same order. It is done by adding or subtracting corresponding elements. The result gives another matrix which has the same order.

Now, the ways of adding and subtracting the matrices are described as follows:

Consider matrices

$$\boldsymbol{A}=\begin{pmatrix} a & c \\ b & d \end{pmatrix} \text{ and } \boldsymbol{B}=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

a) ***Addition of matrices***

$$\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix} a & c \\ b & d \end{pmatrix}+\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\therefore \boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix} a+e & c+f \\ b+g & d+h \end{pmatrix}$$

Properties of matrix addition

i) The sum of two matrices, say $\boldsymbol{A}$ and $\boldsymbol{B}$ with the same order $m\times n$ results to the matrix of the same order.

ii) Matrix addition is commutative i.e. $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$; if $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of the same order.

iii) Matrix addition is associative i.e. $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$; if $\boldsymbol{A},\boldsymbol{B}$ and $\boldsymbol{C}$ are matrices of the same order.

iv) $\boldsymbol{A}+(-\boldsymbol{A})$ or simply $\boldsymbol{A}-\boldsymbol{A}$ matrices gives $m\times n$ zero matrix if $\boldsymbol{A}$ is an $m\times n$ matrix.

Any non-zero matrix added to a zero matrix the sum is non-zero matrix i.e. $\boldsymbol{A}+0=0+\boldsymbol{A}=\boldsymbol{A}$.

b) Subtraction of matrices

Using the same matrices $\boldsymbol{A}$ and $\boldsymbol{B}$;

$$\boldsymbol{A}-\boldsymbol{B}=\begin{pmatrix} a & c \\ b & d \end{pmatrix}-\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\therefore \boldsymbol{A}-\boldsymbol{B}=\begin{pmatrix} a-e & c-f \\ b-g & d-h \end{pmatrix}$$

Properties of matrix subtraction

Matrices subtraction of the same order have the following properties:

i) Subtraction is not commutative *i.e.* $\boldsymbol{A}-\boldsymbol{B}\neq\boldsymbol{B}-\boldsymbol{A}$.

ii) The difference of two matrices results to the matrix of the same order $m\times n$.

iii) Matrix subtraction is not associative, *i.e.* $\boldsymbol{A}-(\boldsymbol{B}-\boldsymbol{C})\neq(\boldsymbol{A}-\boldsymbol{B})-\boldsymbol{C}$.

iv) $\boldsymbol{A}-\boldsymbol{A}$ matrix gives $m\times n$ zero matrices if $\boldsymbol{A}$ is also an $m\times n$ matrix.

v) Any non-zero matrix subtracting a zero matrix, the result is non-zero matrix *i.e.* $\boldsymbol{A}-0=\boldsymbol{A}$.

vi) A zero matrix subtracting any non-zero matrix, the result is negative non-zero matrix i.e $0-\boldsymbol{A}=-\boldsymbol{A}$.

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Example 10.4: Given the matrices

$$A=\begin{pmatrix}3 & 1 & 5\\ 2 & 5 & 7\end{pmatrix},\quad B=\begin{pmatrix}7 & 1 & 2\\ 8 & 9 & 3\end{pmatrix} \text{ and } C=\begin{pmatrix}1 & 1 & 1\\ 4 & 2 & 5\end{pmatrix}$$

Find: a) $A+B$ b) $A-C$ c) $A+B-C$

Solution

a) $A+B=\begin{pmatrix}3 & 1 & 5\\ 2 & 5 & 7\end{pmatrix}+\begin{pmatrix}7 & 1 & 2\\ 8 & 9 & 3\end{pmatrix}$

$=\begin{pmatrix}3+7 & 1+1 & 5+2\\ 2+8 & 5+9 & 7+3\end{pmatrix}$

$\therefore\ A+B=\begin{pmatrix}10 & 2 & 7\\ 10 & 14 & 10\end{pmatrix}$

b) $A-C=\begin{pmatrix}3 & 1 & 5\\ 2 & 5 & 7\end{pmatrix}-\begin{pmatrix}1 & 1 & 1\\ 4 & 2 & 5\end{pmatrix}$

$=\begin{pmatrix}3-1 & 1-1 & 5-1\\ 2-4 & 5-2 & 7-5\end{pmatrix}$

$=\begin{pmatrix}2 & 0 & 4\\ -2 & 3 & 2\end{pmatrix}$

$\therefore\ A-C=\begin{pmatrix}2 & 0 & 4\\ -2 & 3 & 2\end{pmatrix}$

c) $A+B-C=\begin{pmatrix}3 & 1 & 5\\ 2 & 5 & 7\end{pmatrix}+\begin{pmatrix}7 & 1 & 2\\ 8 & 9 & 3\end{pmatrix}-\begin{pmatrix}1 & 1 & 1\\ 4 & 2 & 5\end{pmatrix}$

$=\begin{pmatrix}10 & 2 & 7\\ 10 & 14 & 10\end{pmatrix}-\begin{pmatrix}1 & 1 & 1\\ 4 & 2 & 5\end{pmatrix}$

$=\begin{pmatrix}9 & 1 & 6\\ 6 & 12 & 5\end{pmatrix}$

$\therefore A+B-C=\begin{pmatrix}9 & 1 & 6\\ 6 & 12 & 5\end{pmatrix}$

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Example 10.5: If $A=\begin{pmatrix}2 & 7 & -2\\ 3 & 8 & 5\\ 4 & 1 & -3\end{pmatrix}$ and

$B=\begin{pmatrix}6 & -4 & 9\\ 2 & -5 & 8\\ 1 & 3 & 2\end{pmatrix}$, find;

a) $A+B$ b) $A-B$

Solution

a) $A+B=\begin{pmatrix}2 & 7 & -2\\ 3 & 8 & 5\\ 4 & 1 & -3\end{pmatrix}+\begin{pmatrix}6 & -4 & 9\\ 2 & -5 & 8\\ 1 & 3 & 2\end{pmatrix}$

$=\begin{pmatrix}2+6 & 7-4 & -2+9\\ 3+2 & 8-5 & 5+8\\ 4+1 & 1+3 & -3+2\end{pmatrix}$

$=\begin{pmatrix}8 & 3 & 7\\ 5 & 3 & 13\\ 5 & 4 & -1\end{pmatrix}$

$\therefore \quad A+B=\begin{pmatrix}8 & 3 & 7\\ 5 & 3 & 13\\ 5 & 4 & -1\end{pmatrix}$

b) $A-B=\begin{pmatrix}2 & 7 & -2\\ 3 & 8 & 5\\ 4 & 1 & -3\end{pmatrix}-\begin{pmatrix}6 & -4 & 9\\ 2 & -5 & 8\\ 1 & 3 & 2\end{pmatrix}$

$=\begin{pmatrix}2-6 & 7-(-4) & -2-9\\ 3-2 & 8-(-5) & 5-8\\ 4-1 & 1-3 & -3-2\end{pmatrix}$

$\therefore A-B=\begin{pmatrix}-4 & 11 & -11\\ 1 & 13 & -3\\ 3 & -2 & -5\end{pmatrix}$

Exercise 10.2

1. If $A=\begin{pmatrix}3 & 2 & 5\\ 6 & 0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}0 & 7 & 6\\ 2 & 5 & 8\end{pmatrix}$ find;

   a) $A+B$ b) $A-B$

2. Find matrices P and Q if

   $P+Q=\begin{bmatrix}10 & 8\\ 12 & 6\end{bmatrix}$ and $P-Q=\begin{bmatrix}4 & 16\\ 2 & 18\end{bmatrix}$

3. Verify that $(A+B)+C=A+(B+C)$ where;

   $A=\begin{pmatrix}8 & 2\\ 1 & 4\\ 6 & -2\end{pmatrix}$, $B=\begin{pmatrix}-6 & -4\\ 3 & 7\\ 5 & 9\end{pmatrix}$, and

   $C=\begin{pmatrix}-1 & -10\\ -7 & 5\\ 4 & 8\end{pmatrix}$.

4. If $A=\begin{pmatrix}6 & 1 & -1\\ 2 & 3 & 7\\ 4 & -2 & 8\end{pmatrix}$ and

   $B=\begin{pmatrix}-3 & 5 & 7\\ -4 & -10 & -2\\ 9 & 6 & -1\end{pmatrix}$, find matrix C such that $A+B+C$ is the null matrix.

5. Given

   $A=\begin{pmatrix}10 & -2 & -3\\ 8 & 4 & -5\\ 3 & -2 & -5\end{pmatrix}$, $B=\begin{pmatrix}-6 & -8 & 5\\ 2 & 7 & 2\\ 4 & 3 & 1\end{pmatrix}$

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and $C=\begin{pmatrix}3 & 5 & 2\\ 6 & -1 & -2\\ -3 & -8 & 2\end{pmatrix}$.

Find $(A-B)+C$.

Multiplication of a matrix by a scalar

If A is any matrix and p is any scalar (number), then the product pA is the matrix obtained by multiplying each entry of the matrix A by a scalar p. The matrix pA is said to be a scalar multiple of A.

For example, let matrix $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. If matrix A is multiplied by a scalar $p=3$,

then $3A=3\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}3a & 3b\\ 3c & 3d\end{pmatrix}$

Properties of a scalar multiplication

Let r and s be real numbers, A and B be matrices of the same order. Then:

a) $r(A\pm B)=rA\pm rB$

b) $(r\pm s)A=rA\pm sA$

c) $(rs)A=r(sA)=s(rA)$

Example 10.6: If matrix

$A=\begin{pmatrix}0 & 2 & 0\\ 1 & 0 & 3\\ 1 & 1 & 3\end{pmatrix}$ and $B=\begin{pmatrix}1 & 3 & 1\\ 2 & 1 & 3\\ 0 & 1 & 3\end{pmatrix}$,

find

a) $3A+2B$ b) $4(A+B)$

Solution

a) $3A+2B=3\begin{pmatrix}0 & 2 & 0\\ 1 & 0 & 3\\ 1 & 1 & 3\end{pmatrix}+2\begin{pmatrix}1 & 3 & 1\\ 2 & 1 & 3\\ 0 & 1 & 3\end{pmatrix}$

$=\begin{pmatrix}3\times0 & 3\times2 & 3\times0\\ 3\times1 & 3\times0 & 3\times3\\ 3\times1 & 3\times1 & 3\times3\end{pmatrix}+\begin{pmatrix}2\times1 & 2\times3 & 2\times1\\ 2\times2 & 2\times1 & 2\times3\\ 2\times0 & 2\times1 & 2\times3\end{pmatrix}$

$=\begin{pmatrix}0 & 6 & 0\\ 3 & 0 & 9\\ 3 & 3 & 9\end{pmatrix}+\begin{pmatrix}2 & 6 & 2\\ 4 & 2 & 6\\ 0 & 2 & 6\end{pmatrix}$

$\therefore\ 3A+2B=\begin{pmatrix}2 & 12 & 2\\ 7 & 2 & 15\\ 3 & 5 & 15\end{pmatrix}$

b) $4(A+B)=4\left[\begin{pmatrix}0 & 2 & 0\\ 1 & 0 & 3\\ 1 & 1 & 3\end{pmatrix}+\begin{pmatrix}1 & 3 & 1\\ 2 & 1 & 3\\ 0 & 1 & 3\end{pmatrix}\right]$

$=4\begin{bmatrix}1 & 5 & 1\\ 3 & 1 & 6\\ 1 & 2 & 6\end{bmatrix}$

$=\begin{bmatrix}4\times1 & 4\times5 & 4\times1\\ 4\times3 & 4\times1 & 4\times6\\ 4\times1 & 4\times2 & 4\times6\end{bmatrix}$

$\therefore\ 4(A+B)=\begin{bmatrix}4 & 20 & 4\\ 12 & 4 & 24\\ 4 & 8 & 24\end{bmatrix}$

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Exercise 10.3

1. If $A=\begin{pmatrix}4 & 1\\ 2 & 3\end{pmatrix}$ and $B=\begin{pmatrix}7 & 6\\ 5 & 8\end{pmatrix}$, find the value of $2A+3B$.

2. If $A=\begin{pmatrix}5 & 2\\ 3 & 8\end{pmatrix}$ and $B=\begin{pmatrix}7 & -2\\ -3 & 4\end{pmatrix}$, find matrix C such that $4A+5B=-2C$.

3. If $E=\begin{pmatrix}6 & 4 & 5\\ 2 & 1 & 7\\ 8 & 3 & -2\end{pmatrix}$ and

$F=\begin{pmatrix}-4 & 2 & 12\\ -3 & 8 & 4\\ 7 & 10 & 6\end{pmatrix}$, find the value of $2E+5F$.

4. Given, $2\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}8 & b+10\\ 14 & d-12\end{bmatrix}$, find the values of a, b, c and d.

5. If $4t\begin{bmatrix}2\\ 6\end{bmatrix}+3u\begin{bmatrix}-2\\ -4\end{bmatrix}=\begin{bmatrix}20\\ 24\end{bmatrix}$, find the values of t and u.

Multiplication of a matrix by another matrix

Two matrices can be multiplied if and only if the number of columns in the first matrix is equal to the number of rows of the second matrix, and the resulting matrix will have the number of rows of the first matrix and the number of columns of the second matrix.

For example, if matrix A is of order $m\times a$ and matrix B is of order $b\times n$, then multiplication of matrix AB is possible if and only if $a=b$, and the resulting matrix will be of order $m\times n$.

The procedure for multiplying a matrix by another matrix is as follows:

Let matrix $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and $B=\begin{pmatrix}r & s\\ u & v\end{pmatrix}$,

then $AB=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}r & s\\ u & v\end{pmatrix}$

a) Multiply the first row in A by the first column in B i.e. $\begin{pmatrix}a & b\end{pmatrix}\times\begin{pmatrix}r\\ u\end{pmatrix}=ar+bu$ .....1st element obtained.

b) Multiply the first row in A by the second column in B, i.e.

$\begin{pmatrix}a & b\end{pmatrix}\times\begin{pmatrix}s\\ v\end{pmatrix}=as+bv$ .....2nd element obtained.

c) Multiply the second row in A by the first column in B, i.e.

$\begin{pmatrix}c & d\end{pmatrix}\times\begin{pmatrix}r\\ u\end{pmatrix}=cr+du$ .....3rd element obtained.

d) Multiply the second row in A by the second column in B, i.e.

$\begin{pmatrix}c & d\end{pmatrix}\times\begin{pmatrix}s\\ v\end{pmatrix}=cs+dv$ .....4th element obtained.

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Hence,

$$AB=\begin{pmatrix} 1^{st}\text{ element} & 2^{nd}\text{ element} \\ 3^{rd}\text{ element} & 4^{th}\text{ element} \end{pmatrix}$$

$$\therefore\quad AB=\begin{pmatrix} ar+bu & as+bv \\ cr+du & cs+dv \end{pmatrix}$$

Remark: The previous procedure may be applied to all kind of "$m\times n$" matrices

Example 10.7: Given that $A=\begin{pmatrix} 2 & -2 \\ 3 & 5 \end{pmatrix}$ and $B=\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$;

a) Find AB

b) Is $AB = BA$? give comment.

Solution

a) $AB=\begin{pmatrix} 2 & -2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$

$$=\begin{pmatrix} (2\times3)+(-2\times-1) & (2\times2)+(-2\times4) \\ (3\times3)+(5\times-1) & (3\times2)+(5\times4) \end{pmatrix}$$

$$=\begin{pmatrix} 6+2 & 4+(-8) \\ 9+(-5) & 6+20 \end{pmatrix}$$

$$\therefore\quad AB=\begin{pmatrix} 8 & -4 \\ 4 & 26 \end{pmatrix}$$

b) $BA=\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 3 & 5 \end{pmatrix}$

$$=\begin{pmatrix} (3\times2)+(2\times3) & (3\times-2)+(2\times5) \\ (-1\times2)+(4\times3) & (-1\times-2)+(4\times5) \end{pmatrix}$$

$$\therefore BA=\begin{pmatrix} 12 & 4 \\ 10 & 22 \end{pmatrix}$$

Now, since $AB=\begin{pmatrix} 8 & -4 \\ 4 & 26 \end{pmatrix}$ and $BA=\begin{pmatrix} 12 & 4 \\ 10 & 22 \end{pmatrix}$ then, $AB \neq BA$.

Therefore, multiplication of matrices is not commutative.

Example 10.8: If matrix $A=\begin{pmatrix} 2 & 5 & 7 \\ 8 & 10 & 9 \end{pmatrix}$ and $B=\begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix}$, find AB.

Solution

$$AB=\begin{pmatrix} 2 & 5 & 7 \\ 8 & 10 & 9 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix}$$

From

$$=\begin{pmatrix} (2\times7)+(5\times3)+(7\times2) \\ (8\times7)+(10\times3)+(9\times2) \end{pmatrix}$$

$$=\begin{pmatrix} 14+15+14 \\ 56+30+18 \end{pmatrix}$$

$$\therefore\quad AB=\begin{pmatrix} 43 \\ 104 \end{pmatrix}$$

Since matrix A is of order 2×3 and matrix B is of order 3×1, then matrix AB will be of order 2×1.

Example 10.9: The distribution of number of grades A, B and C and grade points from the departments of Physics, Chemistry, and Biology are represented by matrices P and Q, respectively.

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The matrices are defined as:

$$P = \begin{array}{c} \\ \text{Physics} \\ \text{Chemistry} \\ \text{Biology} \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 20 & 25 & 30 \\ 30 & 18 & 25 \\ 24 & 20 & 35 \end{bmatrix} \end{array} \text{ and}$$

$$Q = \begin{array}{c} A \\ B \\ C \end{array} \begin{bmatrix} 15 \\ 10 \\ 6 \end{bmatrix}$$

Find and interpret the matrix PQ.

Solution

$$\text{Given } P = \begin{bmatrix} 20 & 25 & 30 \\ 30 & 18 & 25 \\ 24 & 20 & 35 \end{bmatrix} \text{ and}$$

$$Q = \begin{bmatrix} 15 \\ 10 \\ 6 \end{bmatrix}$$

then,

$$PQ = \begin{bmatrix} 20 & 25 & 30 \\ 30 & 18 & 25 \\ 24 & 20 & 35 \end{bmatrix} \begin{bmatrix} 15 \\ 10 \\ 6 \end{bmatrix}$$

$$PQ = \begin{bmatrix} 20\times15+25\times10+30\times6 \\ 30\times15+18\times10+25\times6 \\ 24\times15+20\times10+35\times6 \end{bmatrix}$$

$$PQ = \begin{bmatrix} 730 \\ 780 \\ 770 \end{bmatrix}$$

Therefore $PQ = \begin{bmatrix} 730 \\ 780 \\ 770 \end{bmatrix}$ meaning that 730, 780 and 770 are the total points from the departments of Physics, Chemistry, and Biology, respectively.

Exercise 10.4

1. Given two matrices $A = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 7 & 8 \\ 8 & 7 & 7 \end{bmatrix}$

and $B = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 7 & 7 \\ 1 & 2 & 8 \end{bmatrix}$

a) Find AB b) Find BA

c) Is $AB = AB$? Give reason.

2. Given the matrix $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, show that $P^2 - 3P$ is a null matrix.

3. The book seller recorded the number of Basic Applied Mathematics (BAM) and General Studies (GS) books sold for the months of June and July 2017 as follows:

Table 10.2 Number of books for BAM and GS sold in June and July 2017

	June	July
BAM	60	80
GS	30	100

(a) Write this information as a matrix M

(b) If a Basic Applied Mathematics book costs Tsh 12,000 and General Studies book costs Tsh 10,000, re-write these costs as a 2×1 matrix N. What is the defined product between matrices M and N?

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4. If $A=\begin{bmatrix}3 & -5\\ -4 & 2\end{bmatrix}$, show that $A^2-5A=14I$ where I is an identity matrix.

5. The matrix $A=\begin{bmatrix}4 & 2\\ 8 & 6\end{bmatrix}$ represents the number of instruments M and S reproduced per day by factories R and P as seen in Table 10.3.

Table 10.3 Number of instruments M and S reproduced per day by factories R and P.

	Factory R	Factory P
Instrument M	4 units / day	2 units / day
Instrument N	8 units / day	6 units / day

Let $B=\begin{pmatrix}5\\ 6\end{pmatrix}$ represents the number of days the two factories operate per week, i.e, R operates 5 days per week and S operates 6 days per week. Find AB and state what it represents.

6. Let $L=\begin{pmatrix}3 & 5 & 5\\ 7 & 8 & 3\end{pmatrix}$ and $N=\begin{pmatrix}0 & 3\\ 2 & 1\\ 6 & 2\end{pmatrix}$. Find LN and NL. Comment on the answers obtained.

7. If $P=\begin{pmatrix}3 & -5\\ -4 & 5\end{pmatrix}$ and $M=\begin{pmatrix}a\\ b\end{pmatrix}$ such that $PM=\begin{pmatrix}6\\ 9\end{pmatrix}$. Find a and b.

Transpose of a matrix

Let A be $m\times n$ matrix, then the $n\times m$ matrix obtained by interchanging the rows and columns of matrix A is called the transpose of matrix A and it is denoted by A^T.

For instances, if $A=\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$, then

$$A^T=\begin{pmatrix}a & d & g\\ b & e & h\\ c & f & i\end{pmatrix}$$

If $B=\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}$, then $B^T=\begin{pmatrix}2 & 4\\ 3 & 5\end{pmatrix}$

If $D=\begin{pmatrix}3 & 1 & 4\\ 5 & 7 & 8\\ 9 & 2 & 0\end{pmatrix}$, then $D^T=\begin{pmatrix}3 & 5 & 9\\ 1 & 7 & 2\\ 4 & 8 & 0\end{pmatrix}$

Example 10.10: Given that $A=\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}$ and $B=\begin{pmatrix}-4 & 3\\ -5 & 2\end{pmatrix}$, verify that $(AB)^T=B^TA^T$.

Solution: $AB=\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}\begin{pmatrix}-4 & 3\\ -5 & 2\end{pmatrix}$

$$=\begin{pmatrix}-12-5 & 9+2\\ -20-10 & 15+4\end{pmatrix}=\begin{pmatrix}-17 & 11\\ -30 & 19\end{pmatrix}$$

Since, $AB=\begin{pmatrix}-17 & 11\\ -30 & 19\end{pmatrix}$, then

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$$(AB)^T = \begin{pmatrix} -17 & -30 \\ 11 & 19 \end{pmatrix}$$

From $(AB)^T = \begin{pmatrix} -17 & -30 \\ 11 & 19 \end{pmatrix}$

then, $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $A^T = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$

$B = \begin{pmatrix} -4 & 3 \\ -5 & 2 \end{pmatrix}$ and $B^T = \begin{pmatrix} -4 & -5 \\ 3 & 2 \end{pmatrix}$

Thus, $B^T A^T = \begin{pmatrix} -4 & -5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} -12-5 & -20-10 \\ 9+2 & 15+4 \end{pmatrix}$

Therefore, $B^T A^T = \begin{pmatrix} -17 & -30 \\ 11 & 19 \end{pmatrix}$

Hence, verified since $L.H.S = R.H.S$.

Exercise 10.5

1. State the order and find the transpose of each of the following matrices;

   a) $A = \begin{pmatrix} 3 & 2 & 5 \\ 6 & 0 & 1 \end{pmatrix}$ b) $B = \begin{pmatrix} 2 & 0 \\ 3 & 1 \\ 7 & 2 \end{pmatrix}$ c) $C = \begin{pmatrix} 7 & 1 \\ 8 & 2 \end{pmatrix}$

2. Find the transpose of each of the following matrices

   a) $A = \begin{pmatrix} 2 & 7 & -9 \\ 1 & 8 & -2 \\ 3 & 2 & 0 \end{pmatrix}$ b) $B = \begin{pmatrix} 6 & 5 & -5 \\ 2 & 8 & -7 \end{pmatrix}$ c) $C = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 0 & 5 \\ 4 & 4 & 7 \end{pmatrix}$

3. Let $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ 4 & 2 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 3 & 1 & 1 \\ 6 & 2 & 1 \\ 1 & 5 & 4 \end{pmatrix}$. Find $(AR)^T$.

4. If $D = \begin{pmatrix} -2 & 4 & 6 \\ 10 & 14 & 18 \\ -4 & 2 & 2 \end{pmatrix}$ and $E = \begin{pmatrix} -8 & 2 & -10 \\ 2 & 4 & 0 \\ 2 & 6 & 2 \end{pmatrix}$, verify that $(D+E)^T = D^T + E^T$

5. If $G = \begin{bmatrix} 16 & 10 & -6 \\ -8 & 2 & -4 \end{bmatrix}$ and $H = \begin{pmatrix} 2 & 6 & 3 \\ 4 & 8 & 5 \\ 1 & -2 & -1 \end{pmatrix}$, compute $(GH)^T$.

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10.3 Determinant of a matrix

A number that represents the magnitude of a square matrix is called a determinant of a matrix. Determinant of a matrix A is denoted by $\det(A)$ or $|A|$. It can be calculated provided that A is a square matrix.

If $\det(A)=0$ then the matrix A is called singular matrix and if $\det(A)\neq 0$, then the matrix is called non-singular matrix.

a) ***Determinant of a*** 2×2 ***matrix***

Let A be a 2×2 matrix given by

$$A=\begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

then, $|A|=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc.$

Therefore, $\det(A)=ad-bc.$

That is, the determinant of a matrix is obtained by subtracting the product of the entries on the main diagonal from the product of entries in the leading diagonal.

Example 10.11: Find the determinant of matrix A, where $A=\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$.

Solution: Given $A=\begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$.

$$|A|=(4\times 5)-(3\times 2)$$
$$=20-6$$
$$\therefore |A|=14$$

Exercise 10.6

1. If $A=\begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$, find $|A|$
2. If $B=\begin{bmatrix} 2a & 4 \\ 8 & a \end{bmatrix}$, find the possible values of a if B is a singular matrix.
3. Find the value of k if $\begin{vmatrix} 2 & 6 \\ 4 & 8 \end{vmatrix}=\begin{vmatrix} k & 10 \\ 2k & 40 \end{vmatrix}$
4. Given that $L=\begin{bmatrix} 2 & 6 \\ 4 & 8 \end{bmatrix}$, show that $|L|=|L^T|$.
5. Show that $\begin{vmatrix} x^2+4 & x+2 \\ x-2 & -3 \end{vmatrix}=-4(x^2+2)$.

b) ***Determinant of a*** 3×3 ***matrix***

To be able to find the determinant of a 3×3 matrix, the knowledge of minors and cofactors is neccessasry.

i) Minors

The minor of an element is the determinant of a sub-matrix formed by deleting the rows and columns in a given matrix. Each element in a 3×3 matrix has its own minor. The minors may be denoted by M_{ij} where i and j are the numbers of rows and columns, respectively, i.e. i for i^{th} row and j for j^{th} column. For example, M_{13} is a minor of 1^{st} row and 3^{rd} column.

The procedures for obtaining the minors are as follows:

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Let matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Then, let the minors of matrix A be M_{ij}

For M delete the first row and the first column.

$$M_{11} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$
$$= (a_{22} \times a_{33}) - (a_{23} \times a_{32})$$

For M_{12}, delete the first row and the second column.

$$M_{12} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$
$$= (a_{21} \times a_{33}) - (a_{23} \times a_{31})$$

For M_{13}, delete the first row and the third column.

$$M_{13} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= (a_{21} \times a_{32}) - (a_{22} \times a_{31})$$

For M_{21}, delete the second row and the first column.

$$M_{21} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$
$$= (a_{12} \times a_{33}) - (a_{13} \times a_{32})$$

For M_{22}, delete the second row and the second column.

$$M_{22} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$
$$= (a_{11} \times a_{33}) - (a_{31} \times a_{13})$$

For M_{23} delete the second row and the third column

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= (a_{11} \times a_{32}) - (a_{31} \times a_{12})$$

For M_{31}, delete the third row and the first column.

$$M_{31} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$
$$= (a_{12} \times a_{23}) - (a_{13} \times a_{22})$$

For M_{32}, delete the third row and the second column.

$$M_{32} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$
$$= (a_{11} \times a_{23}) - (a_{13} \times a_{21})$$

For M_{33}, delete the third row and the third column.

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$
$$= (a_{11} \times a_{22}) - (a_{12} \times a_{21})$$

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Example 10.12: *Find the* minor of element a_{11} and element a_{23} *in the* matrix

$$\boldsymbol{B}=\begin{pmatrix}2 & 1 & 4\\ 0 & -3 & 5\\ 6 & 7 & 9\end{pmatrix}.$$

Solution

The minor of a_{11}:

$$a_{11}=\begin{vmatrix}2 & 1 & 4\\ 0 & -3 & 5\\ 6 & 7 & 9\end{vmatrix}=\begin{vmatrix}-3 & 5\\ 7 & 9\end{vmatrix}$$

$$=\begin{vmatrix}-3 & 5\\ 7 & 9\end{vmatrix}$$

$$=-3(9)-7(5)$$

$$=-27-35$$

$$\therefore\ a_{11}=-62$$

Therefore, the minor of a_{11} is -62

The minor of a_{23}

$$a_{23}=\begin{vmatrix}2 & 1 & 4\\ 0 & -3 & 5\\ 6 & 7 & 9\end{vmatrix}=\begin{vmatrix}2 & 1\\ 6 & 7\end{vmatrix}$$

$$=\begin{vmatrix}2 & 1\\ 6 & 7\end{vmatrix}$$

$$=2(7)-6(1)$$

$$=14-6$$

$$\therefore\ a_{23}=8$$

Therefore, the minor of a_{23} is 8

ii) Co-factors

A co-factor of an element a_{ij} is defined by multiplying $(-1)^{i+j}$ in each minor of element a_{ij} where i and j represent row and column in which the respective element lies. Co-factor of a matrix $\boldsymbol{A}$ is abbreviated as *Cof (A)*.

For example, if

$$\boldsymbol{A}=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

The co-factor of

$$a_{11}=(-1)^{1+1}\times minor\ of\ a_{11}$$

$$=minor\ of\ a_{11}$$

The co-factor of

$$a_{12}=(-1)^{1+2}\times minor\ of\ a_{12}$$

$$=(-1)\times minor\ of\ a_{12}$$

The co-factor of

$$a_{13}=(-1)^{1+3}\times minor\ of\ a_{13}$$

$$=minor\ of\ a_{13}$$

The co-factor of

$$a_{21}=(-1)^{2+1}\times minor\ of\ a_{21}$$

$$=(-1)\times minor\ of\ a_{21}$$

The co-factor of

$$a_{22}=(-1)^{2+2}\times minor\ of\ a_{22}$$

$$=minor\ of\ a_{22}$$

The co-factor of

$$a_{23}=(-1)^{2+3}\times minor\ of\ a_{23}$$

$$=(-1)\times minor\ of\ a_{23}$$

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The co-factor of

$a_{31} = (-1)^{3+1} \times$ minor of a_{31}

$=$ minor of a_{31}

The co-factor of

$a_{32} = (-1)^{3+2} \times$ minor of a_{32}

$= (-1) \times$ **minor of** a_{32}

The co-factor of

$a_{33} = (-1)^{3+3} \times$ minor of a_{33}

$=$ **minor of** a_{33}

Therefore, the co-factor of any element is equal to -1 or +1 times its minor depending on the position of the respective element. The pattern of signs below may be useful in calculating co-factors.

i.e. $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

Example 10.13: Find the co-factors of element b_{22} and element b_{32} in matrix $\boldsymbol{B}$ below:

$$\boldsymbol{B} = \begin{pmatrix} -3 & 2 & 1 \\ 1 & 4 & 2 \\ 2 & 0 & -2 \end{pmatrix}$$

Solution

Given $\boldsymbol{B} = \begin{pmatrix} -3 & 2 & 1 \\ 1 & 4 & 2 \\ 2 & 0 & -2 \end{pmatrix}$

$$\text{Minor of } b_{22} = \begin{vmatrix} -3 & 2 & 1 \\ 1 & 4 & 2 \\ 2 & 0 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} = (-3 \times -2) - (2 \times 1) = 4$$

$$= (-3 \times -2) - (2 \times 1) = 4$$

$$\therefore \text{ Co-factor of } b_{22} = 4(-1)^{2+2} = 4(1) = 4$$

$$\text{Minor of } b_{32} = \begin{vmatrix} -3 & 2 & 1 \\ 1 & 4 & 2 \\ 2 & 0 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= (-3 \times 2) - (1 \times 1) = -7$$

$$\therefore \text{ Co-factor of } b_{32} = -7(-1)^{3+2} = -7(-1) = 7$$

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Exercise 10.7

1. Find the co-factor of a_{13} in the matrix $\begin{bmatrix} 8 & 2 & 6 \\ 4 & 1 & 5 \\ 9 & 3 & -2 \end{bmatrix}$.

2. Find the minor of a_{31} in the matrix $\begin{bmatrix} -5 & -6 & 1 \\ -2 & 8 & 5 \\ 3 & 7 & 6 \end{bmatrix}$.

3. Find the minor of each element in the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 6 \\ 3 & 1 & 3 \end{bmatrix}$.

4. Find the co-factor of each element in the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 6 \\ 3 & 1 & 3 \end{bmatrix}$.

5. Given $\begin{bmatrix} \mathbf{3} & \mathbf{8} & \mathbf{-12} \\ \mathbf{2} & \mathbf{4} & \mathbf{-10} \\ \mathbf{-6} & \mathbf{5} & \mathbf{-9} \end{bmatrix}$, find the co-factors of element -6 and -10.

Finding determinants of 3×3 matrices using co-factors

The procedures below may be used to find the determinant of 3×3 matrix as follows:

a) Choose any row or any column in the given matrix.

b) Calculate the co-factor of each element in the chosen row or column.

c) The determinant is equal to the sum of the products of the element and its co-factor.

Consider matrix $\boldsymbol{A}$ below:

$$\boldsymbol{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Let us choose row 1 with elements a_1, a_2 and a_3. According to the patterns of signs (+ **or** −) for calculating co-factors;

a_1 lies on positive sign

a_2 **lies on negative sign**

a_3 lies on positive sign

Now, co-factor of $a_1 = +\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$

Co-factor of $a_2 = -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$

Co-factor of $a_3 = +\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

Therefore,

$$|\boldsymbol{A}| = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

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If column 1 is chosen with elements a_1, b_1 and c_1; then, we have:

$$\text{Co-factor of } a_1 = +\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$$

$$\text{Co-factor of } b_1 = -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$$

$$\text{Co-factor of } c_1 = +\begin{vmatrix} a_1 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

Therefore,

$$|A| = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

Example 10.14: Find the determinant of matrix A, where $A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & -3 & 7 \\ 1 & 3 & 8 \end{pmatrix}$

Solution

By choosing row 1;

$$|A| = 3\begin{vmatrix} -3 & 7 \\ 3 & 8 \end{vmatrix} - 4\begin{vmatrix} 2 & 7 \\ 1 & 8 \end{vmatrix} + 5\begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix}$$

$$= 3(-24-21) - 4(16-7) + 5(6+3)$$

$$= 3(-45) - 4(9) + 5(9)$$

$$= -135 - 36 + 45$$

$$= -126$$

$$\therefore |A| = -126$$

Alternatively:

By choosing column 1:

$$= 3(-24-21) - 2(32-15) + 1(28+15)$$

$$= 3(-45) - 2(17) + 43$$

$$|A| = -126$$

$$\therefore |A| = -126$$

Exercise 10.8

1. Find the value of a if $\begin{vmatrix} 1 & a & 0 \\ 2 & 1 & 3 \\ a & 0 & 4 \end{vmatrix} = 10$

2. Find the determinant of the following matrices:

   a) $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 0 & 4 \end{pmatrix}$

   b) $C = \begin{pmatrix} 1 & -2 & 6 \\ 2 & 1 & -3 \\ 3 & 0 & 5 \end{pmatrix}$

   c) $D = \begin{pmatrix} 3 & 5 \\ 4 & -2 \end{pmatrix}$

   d) $F = \begin{pmatrix} -5 & 3 \\ 1 & 4 \end{pmatrix}$

3. If $A = \begin{pmatrix} 3 & 2 & 6 \\ 2 & 1 & 2 \\ -1 & 2 & 0 \end{pmatrix}$, show that $\det(A) = \det(A^T)$.

4. Find the value of y for which the matrix $\begin{pmatrix} -2 & 1 & 1 \\ 3 & 2 & 2 \\ 1 & y & 4 \end{pmatrix}$ is singular.

5. If $C = \begin{bmatrix} 1 & -4 & 11 \\ -2 & -5 & 39 \\ -3 & 2 & 1 \end{bmatrix}$, verify that $\det(C) = \det(C^T)$.

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10.4 Identity and inverse of a matrix

The inverse of a matrix

The inverse of a matrix A is denoted by A^{-1}. The inverse of matrix can only be defined if the given matrix is a square matrix.

Properties of inverse of matrices

Let A and B be matrices with inverse A^{-1} and B^{-1}, respectively. Then the following properties hold:

a) $AA^{-1} = A^{-1}A = I$, where I is identity matrix.

b) $\left(A^{-1}\right)^{-1} = A$

c) The inverse of matrix A exists if and only if A is a non-singular matrix, i.e. $|A| \neq 0$

d) When $|A| = 0$; then, A is a singular matrix. Hence, $\frac{1}{|A|}$ is undefined and therefore singular matrices have no inverse.

e) $(AB)^{-1} = B^{-1}A^{-1}$

Inverse of a 2×2 matrix

Let A be a 2×2 matrix, i.e. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, implying that $\det(A) = ad - cb$. Then, the inverse of A is given by

$$A^{-1} = \frac{1}{\det(A)} \times \text{Adj}(A) \text{ where } Adj(A) =$$

Transpose of co-factor's matrix A, i.e.

$$[\text{Cof}(A)]^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \text{ Hence,}$$

$$A^{-1} = \frac{1}{ad-cb}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example 10.15: Find the inverse of the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$.

Solution

From $A = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$.

$$|A| = (2\times 7) - (3\times 4)$$
$$= 14 - 12$$
$$|A| = 2$$

Now, $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 7 & -3 \\ -4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{2} & -\frac{3}{2} \\ -\frac{4}{2} & \frac{2}{2} \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{7}{2} & -\frac{3}{2} \\ -2 & 1 \end{pmatrix}$$

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Exercise 10.9

1. Given that $C=\begin{bmatrix}3 & 6\\ 3 & 5\end{bmatrix}$, find C^{-1}.
2. Find A^{-1} if, $A=\begin{bmatrix}-3 & 4\\ 3 & -6\end{bmatrix}$.
3. Given that $B=\begin{bmatrix}7 & 5\\ 3 & 2\end{bmatrix}$, find B^{-1}.
4. Find A^{-1} if $A=\begin{pmatrix}\sin x & \cos x\\ -\cos x & \sin x\end{pmatrix}$
5. If $W=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}$, show that $W^2-6W+17I=0$. Hence, find W^{-1}

a) *Inverse of a* 3×3 *matrix*

If B is a 3×3 matrix, then B^{-1} is defined as $B^{-1}=\dfrac{\text{Adj}(B)}{|B|}$ where $\text{Adj}(B)$ is the abbreviation of the word adjoint of matrix B and $|B|$ is the determinant of matrix B.

Adjoint of matrix B

Adjoint of matrix B refers to a matrix obtained by transposing the matrix containing co-factors of elements of matrix B. Therefore, adjoint of matrix B is given by: $\text{Adj}(B)=[\text{Cof}(B)]^T$, where $\text{Cof}(B)$ is an abbreviation of the word co-factor of matrix B.

Let $\text{Cof}(B)=\begin{pmatrix}+m_{11} & -m_{12} & +m_{13}\\ -m_{21} & +m_{22} & -m_{23}\\ +m_{31} & -m_{32} & +m_{33}\end{pmatrix}$

where m_{ij} = minors of elements in matrix B.

Now,

$$\text{Adj}(B)=[\text{cof}(B)]^T=\begin{pmatrix}+m_{11} & -m_{21} & +m_{31}\\ -m_{12} & +m_{22} & -m_{32}\\ +m_{13} & -m_{23} & +m_{33}\end{pmatrix}$$

Therefore, the inverse of matrix B is given by $B^{-1}=\dfrac{\text{Adj}(B)}{|B|}$.

Example 10.16:

Given that $A=\begin{pmatrix}2 & 0 & 6\\ 1 & 5 & 0\\ 4 & -1 & 3\end{pmatrix}$, find:

a) $\det(A)$ b) Minor of A c) co-factor of A d) $\text{Adj}(A)$ e) A^{-1}

Solution

a) From $A=\begin{pmatrix}2 & 0 & 6\\ 1 & 5 & 0\\ 4 & -1 & 3\end{pmatrix}$

$$\det(A)=2\begin{vmatrix}5 & 0\\ -1 & 3\end{vmatrix}-0\begin{vmatrix}1 & 0\\ 4 & 3\end{vmatrix}+6\begin{vmatrix}1 & 5\\ 4 & -1\end{vmatrix}$$

$$=2(15-0)-0(3-0)+6(-1-20)$$

$$=30-0-126$$

$$=-96$$

$\therefore \det(A)=-96$

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b) Let $M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$ be a matrix of minors where m_{ij} = minors of elements in matrix

$$A = \begin{pmatrix} 2 & 0 & 6 \\ 1 & 5 & 0 \\ 4 & -1 & 3 \end{pmatrix}.$$

Then,

$m_{11} = \begin{vmatrix} 5 & 0 \\ -1 & 3 \end{vmatrix}$ = minor for element a_{11}

$= \begin{vmatrix} 5 & 0 \\ -1 & 3 \end{vmatrix} = (15-0) = 15$

$m_{12} = \begin{vmatrix} 1 & 0 \\ 4 & 3 \end{vmatrix}$ = minor for element a_{12}

$= \begin{vmatrix} 1 & 0 \\ 4 & 3 \end{vmatrix} = (3-0) = 3$

$m_{13} = \begin{vmatrix} 1 & 5 \\ 4 & -1 \end{vmatrix}$ = minor for element a_{13}

$= \begin{vmatrix} 1 & 5 \\ 4 & -1 \end{vmatrix} = -1-20 = -21$

$m_{21} = \begin{vmatrix} 0 & 6 \\ -1 & 3 \end{vmatrix}$ = minor for element a_{21}

$= \begin{vmatrix} 0 & 6 \\ -1 & 3 \end{vmatrix} = 0+6 = 6$

$m_{22} = \begin{vmatrix} 2 & 6 \\ 4 & 3 \end{vmatrix}$ = minor for element a_{22}

$= \begin{vmatrix} 2 & 6 \\ 4 & 3 \end{vmatrix} = 6-24 = -18$

$m_{23} = \begin{vmatrix} 2 & 0 \\ 4 & -1 \end{vmatrix}$ = minor for element a_{23}

$= \begin{vmatrix} 2 & 0 \\ 4 & -1 \end{vmatrix} = -2-0 = -2$

$m_{31} = \begin{vmatrix} 0 & 6 \\ 5 & 0 \end{vmatrix}$ = minor for element

$= \begin{vmatrix} 0 & 6 \\ 5 & 0 \end{vmatrix} = 0-30 = -30$

$m_{32} = \begin{vmatrix} 2 & 6 \\ 1 & 0 \end{vmatrix}$ = minor for element a_{32}

$= \begin{vmatrix} 2 & 6 \\ 1 & 0 \end{vmatrix} = 0-6 = -6$

$m_{33} = \begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix}$ = minor for element a_{33}

$= \begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} = 10-0 = 10$

Therefore, the matrix of the minors of matrix A is:

$$\text{Minor of } A = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = \begin{pmatrix} 15 & 3 & -21 \\ 6 & -18 & -2 \\ -30 & -6 & 10 \end{pmatrix}$$

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c) The matrix of co-factors is obtained by multiplying the matrix of minors obtained by signs "+" or "−" as shown in the following pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Then, the matrix of co-factors of matrix A is given by:

$$\text{Cof}(A) = \begin{pmatrix} 15 & -3 & -21 \\ -6 & -18 & 2 \\ -30 & 6 & 10 \end{pmatrix}$$

d) Adjoint of a matrix A is obtained by transposing the co-factors of a matrix A, i.e.

$[\text{Cof}(A)]^T$. Therefore,

$$\text{Adj}(A) = \begin{pmatrix} 15 & -6 & -30 \\ -3 & -18 & 6 \\ -21 & 2 & 10 \end{pmatrix}$$

e) The inverse of a 3×3 matrix is determined by dividing each element in the adjoint matrix by the determinant, i.e.

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = \frac{1}{-96}\begin{pmatrix} 15 & -6 & -30 \\ -3 & -18 & 6 \\ -21 & 2 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{15}{96} & \frac{-6}{-96} & \frac{-30}{-96} \\ \frac{-3}{-96} & \frac{-18}{-96} & \frac{6}{-96} \\ \frac{-21}{-96} & \frac{2}{-96} & \frac{10}{-96} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{5}{32} & \frac{1}{16} & \frac{5}{16} \\ \frac{1}{32} & \frac{3}{16} & -\frac{1}{16} \\ \frac{7}{32} & -\frac{1}{48} & -\frac{5}{48} \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} -\frac{5}{32} & \frac{1}{16} & \frac{5}{16} \\ \frac{1}{32} & \frac{3}{16} & -\frac{1}{16} \\ \frac{7}{32} & -\frac{1}{48} & -\frac{5}{48} \end{pmatrix}$$

Exercise 10.10

1. Find the inverse of the following matrices:

a) $A = \begin{pmatrix} 1 & 3 & 4 \\ 6 & 1 & 2 \\ 3 & 0 & 5 \end{pmatrix}$

b) $B = \begin{pmatrix} 1 & 3 & 0 \\ 6 & 0 & 2 \\ -3 & 2 & 7 \end{pmatrix}$

c) $E = \begin{pmatrix} 3 & -1 & 2 \\ -2 & 2 & 2 \\ 4 & 3 & 3 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 3 & 6 \\ 0 & 5 & 7 \end{pmatrix}$

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2. Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 6 & 1 & 0 \\ 4 & -1 & 3 \end{pmatrix}$, show that

$AA^{-1} = I$.

3. If $D = \begin{pmatrix} 1 & 0 & \cos x \\ 0 & \cos x & 1 \\ 0 & -\sin^2 x & \cos x \end{pmatrix}$, find D^{-1}.

4. Given that $P = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 3 & 1 \\ 0 & 5 & 2 \end{pmatrix}$ and

$V = \begin{pmatrix} 3 & 2 & 5 \\ 0 & 4 & 1 \\ 5 & -2 & 3 \end{pmatrix}$. If P and V are non singular matrices show that $(PV)^{-1} \neq P^{-1}V^{-1}$

5. If $M = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, show that

$M^{-1} = M^2$

10.5 Matrix solution of system of linear equations

There are two methods of solving system of linear equations, namely the inverse matrix method and Cramer's rule.

a) *Solving system of linear eq ations by the inverse matrix method*

i) A system of two equations with two unknowns

Lets consider the system of linear equations below

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases} \quad (10.1)$$

In the matrix form, the system (10.1) becomes;

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (10.2)$$

Let $A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, then, equation (10.2) becomes;

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (10.3)$$

Pre-multiply equation (10.3) by the inverse of A throughout.

That is, $A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ but

$A^{-1}A = I$, identity matrix

Hence, $I\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ whereby

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

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Example 10.17: A woman owns 21 domestic animals. Each of her domestic animals is either a goat or a chicken. If the domestic animals have the total of 76 legs, how many goats and how many chicken does the woman own?

Solution:

Let x be the number of goats the woman owns and y be the number of chickens the woman owns. Then we have:

$$x+y=21 \text{ and } 4x+2y=76$$

Now, find the solution of the system of equations 10.4 by using inverse matrix.

$$\begin{cases} x+y=21 \\ 4x+2y=76 \end{cases} \quad (10.4)$$

Then, in matrix form, the system of equations (10.4) becomes:

$$\begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 21 \\ 76 \end{pmatrix} \quad (10.5)$$

Let matrix $\begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}$ from the system (10.5) be $\boldsymbol{A}$, that is, $\boldsymbol{A}=\begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}$, then $det(\boldsymbol{A})=-2$.

The inverse of the matrix $\boldsymbol{A}$, is;

$$\boldsymbol{A}^{-1}=\frac{1}{det(\boldsymbol{A})}Adj(\boldsymbol{A})$$

$$=-\frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 1 \end{pmatrix}$$

Multiply $\boldsymbol{A}^{-1}$ on both sides of equation (10.5).

$$-\frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=-\frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 21 \\ 76 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=-\frac{1}{2}\begin{pmatrix} -34 \\ -8 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 17 \\ 4 \end{pmatrix} \quad (10.6)$$

Therefore, from equation (10.4) the value of $x=17$ and $y=4$; which implies that the woman has 17 goats and 4 chickens.

Example 10.18: Solve the following system of equations by using the inverse matrix:

$$\begin{cases} 3x-4y=1 \\ x+2y=7 \end{cases}$$

Solution: In matrix form, the system becomes:

$$\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}=\begin{pmatrix} 1 \\ 7 \end{pmatrix} \quad (10.7)$$

Let $\boldsymbol{A}=\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$

Thus, $|\boldsymbol{A}|=3(2)-(1\times -4)$

$=6+4$

$=10$

Now, since $\boldsymbol{A}^{-1}=\frac{1}{det(\boldsymbol{A})}Adj(\boldsymbol{A})$

Then, $\boldsymbol{A}^{-1}=\frac{1}{10}\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$

The equation (10.7) is presented as follows:

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$$A\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1\\7\end{pmatrix} \quad (10.8)$$

Pre-multiply by A^{-1} on both sides of equation (10.8).

$A\begin{pmatrix}x\\y\end{pmatrix}=A^{-1}\begin{pmatrix}1\\7\end{pmatrix}$ but $AA^{-1}=I$ where

I is an identity matrix.

$$I\begin{pmatrix}x\\y\end{pmatrix}=A^{-1}\begin{pmatrix}1\\7\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{10}\begin{pmatrix}2&4\\-1&3\end{pmatrix}\begin{pmatrix}1\\7\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{10}\begin{pmatrix}2+28\\-1+21\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{10}\begin{pmatrix}30\\20\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{30}{10}\\\frac{20}{10}\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$$

$\therefore\ x=3$ and $y=2$

Exercise 10.11

1. Use the inverse matrix to solve the system of equations below

$$\begin{cases}2x-3y=7\\x+4y=9\end{cases}$$

2. Solve the following system of linear equation by using the matrix method

$$\begin{cases}2x+y=4\\x+3y=7\end{cases}$$

3. Using inverse matrix, solve the following system of linear equation

$$\begin{cases}3x-5y=8\\x+3y=12\end{cases}$$

4. The sum of ages of the father and his son is 52 years. Eight years ago the age of the father was 5times that of the son. By using the inverse matrix method, find the age of the father and his son.

5. A business person is running a grocery near your school. He is selling pieces of cake and soda. Each piece of cake costs Tsh 1,500 and each bottle of soda costs Tsh 500. In one evening, the person made a total of Tsh 78,500 by selling a total of 87 pieces of cake and soda. How many pieces of cake and bottles of soda were sold?

v) A system of three equations with three unknowns

Consider the following system of linear equations:

$$\begin{cases}a_1x+b_1y+c_1z=d_1\\a_2x+b_2y+c_2z=d_2\\a_3x+b_3y+c_3z=d_3\end{cases} \quad (10.9)$$

In matrix form, the system (10.9) becomes:

$$\begin{pmatrix}a_1&b_1&c_1\\a_1&b_2&c_2\\a_1&b_3&c_3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix} \quad (10.10)$$

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Let the matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_1 & b_2 & c_2 \\ a_1 & b_3 & c_3 \end{pmatrix}$ be A where matrix A is called the coefficient matrix.

Now, equation (10.10) becomes;

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \qquad (10.11)$$

Pre-multiply equation (10.11) by A^{-1} on both sides.

$$AA^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \text{ but } AA^{-1} = I \text{ where } I \text{ is an identity matrix.}$$

So, $I\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ where $I\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Example 10.19: Solve the following system of linear equations using the inverse matrix:

$$\begin{cases} x+2y+3z=6 \\ 2x+y+z=5 \\ 3x+y-2z=1 \end{cases} \qquad (10.12)$$

Solution

In matrix form, the system (10.12) of linear equations can be written as follows:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$$

Let the coefficient matrix be A, i.e.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix}$$

If $\text{Cof}(A)$ is the matrix of the co-factors, then;

$$\text{Cof}(A) = \begin{pmatrix} +\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}$$

$$\text{Cof}(A) = \begin{pmatrix} -3 & 7 & -1 \\ 7 & -11 & 5 \\ -1 & 5 & -3 \end{pmatrix}$$

Now, the adjoint of matrix A is:

$$\text{Adj}(A) = [\text{Cof}(A)]^T$$

$$\text{Adj}(A) = \begin{pmatrix} -3 & 7 & -1 \\ 7 & -11 & 5 \\ -1 & 5 & -3 \end{pmatrix}$$

and the determinant of A is:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$

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$$= -3 - 2(-7) + 3(-1)$$

$$|A| = 8$$

Then, $A^{-1} = \frac{1}{\det(A)} \text{Adj}(A)$

$$A^{-1} = \frac{1}{8}\begin{pmatrix} -3 & 7 & -1 \\ 7 & -11 & 5 \\ -1 & 5 & -3 \end{pmatrix} \qquad (10.13)$$

Hence, use A^{-1} equation (10.13) to solve the system (10.12).

From $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$ (10.14)

Pre-multiply equation (10.14) throughout by A^{-1}.

$$A^{-1}A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix} \text{ but } AA^{-1} = I$$

whereby I is an identity matrix.

Now, $I\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -3 & 7 & -1 \\ 7 & -11 & 5 \\ -1 & 5 & -3 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} -18 + 35 - 1 \\ 42 - 55 + 5 \\ -6 + 25 - 3 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} 16 \\ -8 \\ 16 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{16}{8} \\ \frac{-8}{8} \\ \frac{16}{8} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$\therefore x = 2,\ y = -1$ and $z = 2$

Exercise 10.12

1. Solve the following system of linear equations by using matrix method

$$\begin{cases} 2x + 3y - 2z = 1 \\ x - 4y + 3z = 2 \\ 4x + y - z = 4 \end{cases}$$

2. Use the inverse matrix to solve the following system of equations

$$\begin{cases} x + 3y - 2z = 5 \\ x - 4y + 3z = -7 \\ x + y - z = 0 \end{cases}$$

3. If $A = \begin{bmatrix} 2 & -6 & 2 \\ 4 & 2 & -8 \\ 12 & -14 & 16 \end{bmatrix}$, find A^{-1}.

Hence use the inverse matrix method to solve the following system of linear equations

$$\begin{cases} 2x - 6y + 2z = -2 \\ 4x + 2y - 8z = -2 \\ 12x - 14y + 16z = 14 \end{cases}$$

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4. A botanist requires 29, 13 and 16 units of magnesium, phosphorus, and potassium, respectively for his nursery. A concentrated product contains 2, 1 and 3 units of magnesium, phosphorus, and potassium per carton, respectively, a diluted product contains 3, 1 and 2 units of magnesium, phosphorus and potassium per carton, respectively and a dry product contains 4, 2 and 1units of magnesium, phosphorus, and potassium per carton, respectively. Formulate and express a system of equations in a matrix form. By using the inverse matrix method, find how many of each should be purchased in order to meet the requirements.

5. In a market survey, three commodities rice, beans and flour were considered. In finding out the index number some fixed weights were assigned to three varieties in each of the commodities. The Table 10.4 provides the information regarding the consumption of three commodities according to three varieties and the total weight received by the commodity.

Table 10.4 Consumption of commodities, varieties and the total weight

Commodity	Variety			Total weight
	I	*II*	*III*	
Rice	9	15	18	81
Beans	6	12	15	63
Flour	3	6	9	33

Find the weights assigned to three varieties by using inverse matrix method given that the weights assigned to a commodity are equal to the sum of weights of the various varieties multiplied by the corresponding consumption.

b) Solving a system of linear equations using the Cramer's rule

Here, two systems of linear equations will be discussed. These are as follows:

i) A system of two equations and two unknowns

Consider the following system of equations:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2 + b_2y = c_2 \end{cases} \qquad (10.15)$$

This can be solved by eliminating y in the system (10.15).

$$-\begin{cases} a_1b_2x + b_1b_2y = c_1b_2 \\ a_2b_1x + b_1b_2y = c_2b_1 \end{cases}$$

$$\frac{x(a_1b_2 - a_2b_1)}{a_1b_2 - a_2b_1} = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$$

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \qquad (10.16)$$

Note that the determinant of

$\begin{pmatrix} c_1 & b_1 \\ c_2 & b_2 \end{pmatrix} = c_1b_2 - c_2b_1$ which is the

numerator expression of equation (10.16) and the determinant of

$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1b_2 - a_2b_1$ which is the

denominator expression of equation (10.16). Therefore equation (10.16) becomes:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

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Similarly, by eliminating x in equation (10.15), it can be shown that $y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$,

and therefore, $y=\dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$.

So, let $A=\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ be the coefficient matrix; then, replacing the first column of matrix A by the values c_1 and c_2 yealds

$$A_1=\begin{pmatrix} c_1 & b_1 \\ c_2 & b_2 \end{pmatrix}$$

and replacing the second column of matrix A by values c_1 and c_2 gives

$$A_2=\begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}.$$

Therefore, a formula for solving a system of linear equations by using determinants is called Cramer's rule and is given by:

$$x=\frac{|A_1|}{|A|} \text{ and } y=\frac{|A_2|}{|A|}$$

Example 10.20: Solve the following system of linear equations by using Cramer's rule.

$$\begin{cases} 2x+3y=9 \\ 4x-2y=10 \end{cases}$$

Solution

From $\begin{cases} 2x+3y=9 \\ 4x-2y=10 \end{cases}$

In the matrix form, it will be:

$$\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 9 \\ 10 \end{pmatrix}$$

Let A be the coefficient matrix

$$A=\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}$$

Then, $A_1=\begin{pmatrix} 9 & 3 \\ 10 & -2 \end{pmatrix}$ where column one is replaced by 9 and 10 and $A_2=\begin{pmatrix} 2 & 9 \\ 4 & 10 \end{pmatrix}$ where column two is replaced by 9 and 10

Now, $|A|=(2\times-2)-(4\times3)$

$=-4-12$

$|A|=-16$

$|A_1|=(9\times-2)-(10\times3)$

$=-18-30=-48$

$|A_2|=(2\times10)-(4\times9)$

$=20-36=-16$

Hence, $x=\frac{|A_1|}{|A|}=\frac{-48}{-16}=3$

$y=\frac{|A_2|}{|A|}=\frac{-16}{-16}=1$

$\therefore$ The value of $x=3$ and $y=1$

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Exercise 10.13

1. Solve the system of linear equations by using Cramer's rule
$$\begin{cases} 2x-4y=6 \\ 3x+4y=9 \end{cases}$$

2. Solve the system of linear equations by using Cramer's rule
$$\begin{cases} 5x+y=18 \\ 2x+3y=15 \end{cases}$$

3. Solve the system of linear equations by using Cramer's rule
$$\begin{cases} 3x+2y=9 \\ 8x-y=5 \end{cases}$$

4. Anna and John went to the market to buy tomatoes and onions. Anna had Tsh 4000 and John had Tsh 5,200. Anna bought 10 tomatoes and 15 onions. John bought 12 tomatoes and 20 onions. How much did they spend for each item?

5. Consider the data on prices (Tsh per kg) of rice and beans in the months of November and December.

Table 10.5 The prices of rice and beans in Tanzanian shillings

	Rice	Beans
November	5,000	1,000
December	2,000	3,000

The family can spend Tsh 18,000 and Tsh 15,000 in November and December, respectively on rice and beans. Now, if the family wants to purchase the same combination of rice and beans in November and December, use Cramer's rule to find how much rice and how much beans the family can buy.

ii) A system of three linear equations with three unknowns

Consider the following system of linear equations:

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases} \qquad (10.17)$$

Write the system of equations (10.17) in matrix form as follows:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \qquad (10.18)$$

Let $\boldsymbol{A}=\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ be a coefficient matrix. By Cramer's rule, the values of x, y, and z are obtained by using the following formulae:

$$x=\frac{|\boldsymbol{A}_1|}{|\boldsymbol{A}|},\ y=\frac{|\boldsymbol{A}_2|}{|\boldsymbol{A}|} \text{ and } z=\frac{|\boldsymbol{A}_3|}{|\boldsymbol{A}|}$$

whereby $|\boldsymbol{A}|$ is the determinant of matrix $\boldsymbol{A}$ which is the coefficient matrix; i.e.

$$|\boldsymbol{A}|=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & a_3 \end{vmatrix}$$

$|\boldsymbol{A}_1|$ is the determinant of matrix $\boldsymbol{A}$, when the first column values are replaced by values d_1, d_2 and d_3. The determinant of the new matrix $\boldsymbol{A}_1$ is given by

$$|\boldsymbol{A}_1|=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$|\boldsymbol{A}_2|$ is the determinant of matrix $\boldsymbol{A}$, when the second column values are replaced by values d_1, d_2 and d_3. The determinant of

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the new matrix A_2 is given by

$$|A_2| = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$|A_3|$ is the determinant of matrix A, when the third column values are replaced by values d_1, d_2 and d_3. The determinant of the new matrix A_3 is given by

$$|A_3| = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Example 10.21: Solve the following system of equations by using Cramer's rule:

$$\begin{cases} 3x+2y+z=7 \\ x+3y-z=7 \\ 2x-y+3z=0 \end{cases}$$

Solution

In the matrix form, the given equations will be:

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix}$$

Let A be the coefficient matrix, i.e.

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

Then, $|A| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

$$= 3\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix}$$

$$= 3(9-1) - 2(3+2) + 1(-1-6)$$

$$= 24 - 10 - 7$$

$$|A| = 7$$

Then, $A_1 = \begin{pmatrix} 7 & 2 & 1 \\ 7 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ when column 1 is replaced by 7, 7, and 0

$$|A_1| = 7\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 7 & -1 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 7 & 3 \\ 0 & -1 \end{vmatrix}$$

$$= 7(9-1) - 2(21-0) + 1(-7-0)$$

$$|A_1| = 7$$

$A_2 = \begin{pmatrix} 3 & 7 & 1 \\ 1 & 7 & -1 \\ 2 & 0 & 3 \end{pmatrix}$ when column 2 is replaced by 7, 7, and 0.

$$|A_2| = 3\begin{vmatrix} 7 & -1 \\ 0 & 3 \end{vmatrix} - 7\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 7 \\ 2 & 0 \end{vmatrix}$$

$$= 3(21-0) - 7(3+2) + 1(0-14)$$

$$|A_2| = 14$$

$A_3 = \begin{pmatrix} 3 & 2 & 7 \\ 1 & 3 & 7 \\ 2 & -1 & 0 \end{pmatrix}$ when column 3 is replaced by 7, 7, and 0.

$$|A_3| = 3\begin{vmatrix} 3 & 7 \\ -1 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 7 \\ 2 & 0 \end{vmatrix} + 7\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix}$$

$$= 3(0+7) - 2(0-14) + 7(-1-6)$$

$$|A_3| = 0$$

Hence, $x = \dfrac{|A_1|}{|A|} = \dfrac{7}{7} = 1$, $y = \dfrac{|A_2|}{|A|} = \dfrac{14}{7} = 2$

and $z = \dfrac{|A_3|}{|A|} = \dfrac{0}{7} = 0$

$\therefore$ The value of $x = 1$, $y = 2$ and $z = 0$

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Exercise 10.14

1. Using Cramer's rule method, solve the following system of simultaneous equations

a) $\begin{cases} x-2y+z=3 \\ 2x+y-z=5 \\ 3x-y+2z=12 \end{cases}$

b) $\begin{cases} 2x+y-z=0 \\ x+z=4 \\ x+y+z=0 \end{cases}$

c) $\begin{cases} 2x+3y-2z=1 \\ x-4y+3z=2 \\ 4x+y-z=4 \end{cases}$

d) $\begin{cases} x+3y-2z=5 \\ x-4y+3z=-7 \\ x+y-z=0 \end{cases}$

2. A manufacturer produces three types of products P_1, P_2 and P_3. Each product is first processed in a machine M_1, secondly in a machine M_2 and then sent to another machine M_3 for finishing. Table 10.6 shows the number of minutes and the total time available.

Table 10.6 The number of minutes and the total time available for producing products P_1, P_2 and P_3

	M_1	M_2	M_3
P_1	12	6	18
P_2	9	12	6
P_3	6	18	9
Total time	180	270	210

Calculate the number of units of three types of products produced by constructing a matrix equation and then solve it by the Cramer's rule.

Revision exercise 10

1. a) Given the matrices $\boldsymbol{A}=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix}$ and $\boldsymbol{B}=\begin{pmatrix} 3 & 2 & 3 \\ -2 & 1 & 1 \\ 5 & 1 & 0 \end{pmatrix}$

Find the following matrices: i) $\boldsymbol{A}+\boldsymbol{B}$ ii) $\boldsymbol{B}-\boldsymbol{A}$

b) Show that the matrix addition in a) above is commutative.

2. If matrix $\boldsymbol{B}=\begin{pmatrix} -2 & -1 & 3 \\ 3 & 1 & 0 \\ 3 & 2 & -2 \end{pmatrix}$. Find the matrix $t\boldsymbol{B}$ given that $t=4$.

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3. If the matrices $P=\begin{pmatrix}-8 & -1 & 3\\ 1 & 0 & 0\\ 3 & 2 & 1\end{pmatrix}$ and $Q=\begin{pmatrix}-2 & -1 & 3\\ 3 & 1 & 0\\ 3 & 2 & -3\end{pmatrix}$. Find the following matrices:

   a) PQ b) P^2

   c) QP and then show that $PQ \neq QP$ by stating the property of matrix multiplication.

4. Given the matrix $A=\begin{pmatrix}-1 & 2 & 3\\ 2 & 1 & 0\\ 3 & 2 & 1\end{pmatrix}$, find the matrix from the expression A^2-2A+I

5. Find the determinant of the following matrices:

   a) $A=\begin{pmatrix}1 & -1 & 7\\ 1 & 5 & 6\\ -2 & 2 & 1\end{pmatrix}$ b) $B=\begin{pmatrix}2 & -1 & 1\\ 1 & 0 & -1\\ 2 & -4 & 1\end{pmatrix}$

6 If $M=\begin{pmatrix}-1 & -1 & 1\\ 1 & 0 & 0\\ 3 & 2 & k\end{pmatrix}$, find the value of k given that matrix M is a singular matrix.

7 If $A=\begin{pmatrix}2 & 1 & 2\\ 2 & 2 & -2\\ 2 & -2 & -1\end{pmatrix}$, show whether or not $A^3=9A$

8. Find the inverse of the matrix $N=\begin{pmatrix}1 & -3 & 0\\ 2 & 0 & 1\\ 4 & 1 & 3\end{pmatrix}$.

9. Use the inverse matrix obtained in question 8 above to solve the following system of simultaneous equations: $\begin{cases}x-3y & =-5\\ 2x & +z=5\\ 4x+y+3z & =15\end{cases}$

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10. Solve the following system of simultaneous equations by matrix method:

$$\begin{cases} 2y + z = 13 \\ 4x - 4y + 5z = 19 \\ 2x + y - z = 1 \end{cases}$$

11. Using Cramer's rule, find the solution of the following system of simultaneous equations:

a) $\begin{cases} 2x + 5y - 3z = 5 \\ x - 4y + z = -5 \\ 4x + 3y - z = 10 \end{cases}$

b) $\begin{cases} 2x + y + z = 5 \\ 3x + y - 2z = 1 \\ x + y - z = 1 \end{cases}$

12. The records of 10 football matches played by three teams; A, B, and C are given in matrix $\boldsymbol{P}$ and the points awarded for a win, draw, and lose matches are given in matrix $\boldsymbol{S}$.

$$\boldsymbol{P} = \begin{pmatrix} \underset{\downarrow}{\text{win}} & \underset{\downarrow}{\text{draw}} & \underset{\downarrow}{\text{lose}} \\ 6 & 3 & 1 \\ 7 & 1 & 2 \\ 5 & 5 & 0 \end{pmatrix} \begin{matrix} \leftarrow \text{Team A} \\ \leftarrow \text{Team B} \\ \leftarrow \text{Team C} \end{matrix} \quad \text{and} \quad \boldsymbol{S} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \begin{matrix} \leftarrow \text{Win} \\ \leftarrow \text{Draw} \\ \leftarrow \text{Lose} \end{matrix}$$

a) Find the product $\boldsymbol{PS}$ and interpret the results

b) State which team leads the league.

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Chapter Eleven

Linear programming

Introduction

Linear programming refers to the analyses of mathematical problems in which a linear function (objective function) of several variables is to be optimised (maximised or minimised) when the variables are subject to a number of constraints in the mathematical linear equations and/or inequalities. Knowledge of linear programming is applicable in various fields such as in business, manufacturing and transportation, investment planning, and optimum allocation of resources in manufacturing of goods, among many other applications. In this chapter, you will learn about identifying real-life problems related to linear programming, linear programming problems, drawing graphs of linear programming inequalities, and finding solutions of linear programming problems.

11.1 Applications of linear programming problems in real life

The following are examples of real life problems where linear programming techniques can be applied:

a) Minimizing the risk in investment portfolio, subject to achieving a certain return. Many investment companies are using optimization and linear programming extensively to decide how to allocate assets.

b) Carpentry and tailoring problems. For example, carpenters and tailors with the knowledge and skills of linear programming, will be able to use properly the scarce resources available for the purpose of producing as many products as possible to generate maximum profit after selling the products.

c) Investment planning problems. An investor may wish to determine the proper allocation of resources among the available investment opportunities by using linear programming techniques.

d) Product-mix problems. The product-mix problems are typical linear programming problems in which there is competition for limited resources among several products. The problem is to allocate the resources to the production in order to maximise profit or minimise cost.

e) Balanced diet problems. A dietician may apply linear programming techniques for proper advice on the minimum nutritional requirements (intake) basing on the contents of types of food available.

f) Manufacturing and transporting problems. For example, manufactured products in the factories or industries are normally stored in warehouses or depots before they are supplied to customers. If the industry wishes to maximize the profit, linear

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programming skills is required in order to minimize the cost of transporting the products from depots to customers.

11.2 Formulation of linear programming problems (LPP)

The formulation of LPP in mathematical model can be achieved by translating the word problem into linear inequalities or equalities. The following key terms in the formulation of linear programming problems are used, namely: decision variables, objective function, constraints, and non-negativity constraints. However, the main parts of linear programming problem are decision variables, objective function and constraints.

a) Decision variables

These are variables such as x and y which represent quantities that need to be determined when solving the linear programming problems.

b) Objective function

This is a linear function used to find the minimum or maximum value of the problem and gives the solution required in maximizing or minimizing linear programming problems. It is written in the form $f(x, y) = ax + by$; where $a, b \in \mathbb{R}$. This is a decision equation that will be satisfied by constraints.

c) Constraints

These are linear inequalities defining limitations on decisions. They arise from various sources such as limited resources, contractual obligations, or physical laws. The inequality signs are $<$, $>$, $\leq$ and $\geq$.

Some words which are used in linear programming problems are shown in Table 11.1.

Table 11. 1 Common words representing inequality symbols

Word	Inequality symbol used
Greatest value, maximum value, at most, not more than	$\leq$
Lowest value, minimum value, at least, not less than	$\geq$

d) Non-negativity constraints

The linear inequalities $x \geq 0$ and $y \geq 0$ are called non-negativity constraints. These are included because the decision variables x and y cannot be negative values. In other words, items or resources cannot take negative values.

Example 11.1: A cook wishes to mix two types of food, F_1 and F_2 in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food F_1 costs Tsh 600 per kilogram and food F_2 costs Tsh 800 per kilogram. Food F_1 contains 3 units per kilogram of vitamin A and 5 units per kilogram of vitamin B, while food F_2 contains 4 units per kilogram of vitamin A and 2 units per kilogram of vitamin B. Formulate a linear programming problem to minimize the cost of the mixture. Table 11.2 shows the summary of the linear programming problem.

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Solution

Table 11.2 Summary of the vitamin contents of food F1 and F2 and their respective costs

Vitamin content	Food in kg		Minimum requirement (in units)
	F_1	F_2	
Vitamin A (units/kg)	3	4	8
Vitamin B (units/kg)	5	2	11
Costs (Tsh/kg)	600	800	

In formulating the LPP based on the Example 11.1, use the following steps:

Step 1. The number of kilograms of food F_1 and F_2 contained in the mixture are the decision variables. Let the mixture contain x kg of food F_1 and y kg of food F_2.

Step 2. In this example, the constraints are the minimum requirements of the vitamins. The minimum requirement of vitamin A is 8 units. Therefore, $3x+4y \geq 8$. Similarly, the minimum requirement of vitamin B is 11 units. Therefore, $5x+2y \geq 11$.

Step 3. The cost of purchasing 1 kg of food F_1 is Tsh 600. The cost of purchasing 1 kg of food F_2 is Tsh 800. The total cost of purchasing x kg of food F_1 and y kg of food F_2 is: $f(x, y) = 600x + 800y$ which is the objective function.

Step 4. The non-negativity conditions are: $x \geq 0$ and $y \geq 0$. Therefore, the mathematical linear programming problem is:

Minimize: $f(x, y) = 600x + 800y$

Subject to: $3x+4y \geq 8$

$5x+2y \geq 11$

$x \geq 0,\ y \geq 0$

Example 11.2: A furniture company produces tables and chairs. The production of each furniture is similar as both require a certain number of hours of carpentry work and a certain number of labour hours in the painting department per week. Each table takes 4 hours of carpentry and 2 hours in the painting department. Each chair takes 3 hours of carpentry and 1 hour in the painting department. During the current stated production period, 240 hours of carpentry time are available and 100 hours of painting time are available. Each table sold yields a profit of Tsh 70,000 and each chair sold yields a profit of Tsh 50,000. Formulate this problem as a linear programming problem to maximize the profit.

Solution

Summarized data from the given linear programming problem is given in Table 11.3.

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Table 11.3 Summary of the time and costs for producing furniture by a furniture company

Departments	Hours required		Available Hours
	Tables	Chairs	
Carpentry	4	3	240
Painting	2	1	100
Profit per item (Tshs.)	70,000	50,000	

The objective is to maximize profit.

The constraints are:

a) The hours of carpentry time used cannot exceed 240 hours per week.

b) The hours of painting time used cannot exceed 100 hours per week.

c) The number of tables and chairs must be non-negative.

Now, the decision variables that represent the actual decision to be made are defined as:

x represents the number of tables to be produced

y represents the number of chairs to be produced

Therefore, the linear programming (LP) problem in terms of x, y and profit (P) is:

Maximize: $P(x, y) = 70000x + 50000y$

Subject to: $4x + 3y \leq 240$

$2x + y \leq 100, \quad x \geq 0 \quad y \geq 0$

Exercise 11.1

1. A certain brewery produces two types of beers, A and B. Suppose that production is limited by scarcity of resources of yeast, hops, and barley malt. To make one crate of beers of type A; 5 kg of yeast, 2 kg of hops and 10 kg of malts are required. To make one crate of beers of type B; 3 kg of yeast, 2 kg of hops, and 7 kg of malts are required. Suppose that only 80 kg of yeast, 60 kg of hops, and 190 kg of malts are available. If the brewery makes a profit of Tsh 13,000 per crate of beer of type A and Tsh 23,000 per crate of beers of type B, formulate this problem as a linear programming problem to maximize the profit.
2. A wheat and barley farmer has 168 hectares of ploughed land and a capital of Tsh 2,000,000. It costs Tsh 60,000 to sow one hectare of wheat and Tsh 80,000 to sow one hectare of Barley. If one hectare of wheat produces a profit of Tsh 600,000 and one hectare of Barley produces a profit of Tsh 500,000, formulate this problem as a linear programming problem to maximize the profit.
3. A cook wishes to mix together two types of foods, I and II, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B, and 8 units of vitamin C. The

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vitamin contents of one kg of food is given in Table 11.4.

Table 11.4 Types of vitamin contents in kilograms of food I and food II

	Vitamin A	Vitamin B	Vitamin C
Food I	1	2	3
Food II	2	2	1

One kg of food I costs Tsh 6,000 and one kg of food II costs Tsh 10,000. Formulate this problem as a linear programming problem to minimize the cost.

4. Rehema can buy two types of plant food, X and Y. Each cubic metre of X contains 20 kg of phosphoric acid, 30 kg of nitrogen, and 5 kg of potash, and each cubic metre of Y contains 10 kg of phosphoric acid, 30 kg of nitrogen, and 10 kg of potash. The minimum monthly requirements are 460 kg of phosphoric acid, 960 kg of nitrogen, and 220 kg of potash. If food X costs Tsh 30,000 per cubic metre and food Y costs Tsh 35,000 per cubic metre, formulate this problem as a linear programming problem to minimize the cost.
5. Victor has exactly 32 units of yellow dye and 54 units of green dye. He plans to mix as many gallons as possible of colour A and colour B. Each gallon of colour A requires 4 units of yellow dye and 1 unit of green dye. Each gallon of colour B requires 1 unit of yellow dye and 6 units of green dye. Formulate this problem as a linear programming problem to maximize the number of gallons.
6. Mwanahamisi wants to earn at least Tsh 30,000 per week. Her manager has agreed to pay her Tsh 6,000 to mow the lawn and Tsh 5,000 to weed the garden-per day. Suppose she mows the lawn at least once. Formulate this problem as a linear programming problem to minimize the number of days she will have to spend weeding the garden and mowing the lawn of her manager's house.
7. Form Four students at Mpirani Secondary School are about to take a test that contains questions of type A and B. Questions of type A worth 10 points each and questions of type B worth 25 points each. They must do at least 3 questions of type A but not more than 12. They must do at least 4 questions of type B but not more than 15. In total they cannot do more than 20 questions. Formulate this problem as a linear programming problem to maximize the scores.

11.3 Graphs of linear inequalities

The graph of a linear inequality of two variables is the set of all points in the xy–plane which satisfies the inequality. The graph of a linear inequality has a region of the plane whose boundary is a straight line. A linear inequality with two variables to be discussed in this section has the following forms:

a) $ax+by \le c$ or

b) $ax+by < c$ or

c) $ax+by \ge c$ or

d) $ax+by > c$, whereby a, b, and c are real numbers and $(a,b) \neq (0,0)$

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Steps for graphing linear inequalities

1. Replace the inequality symbol with an equal sign to form the equation of the boundary line of the graph.
2. Draw the graph of the straight line that is the boundary. Use a dotted line if an equal sign is not included i.e. $>$ or $<$. Use a solid line if equal sign is included i.e. $\leq$ or $\geq$.
3. Choose any convenient test point which does not lie on the boundary line and substitute the coordinates of that point in the inequality.
4. If the inequality is true at the test point, then that is the wanted region and shade it.

Example 11.3: Draw the graph of $2x + y < 4$.

Solution: First, replace the inequality sign with an equal sign, i.e. $2x + y = 4$.

Then this equation forms the boundary line of the graph of the original inequality. Since the inequality has $<$ sign, then the line will be dotted.

Plot the graph using the given equation i.e. $2x + y = 4$

Table 11.5 The **x** and **y** intercepts for $2x + y = 4$

x	0	2
y	4	0

The graph is presented in Figure 11.1

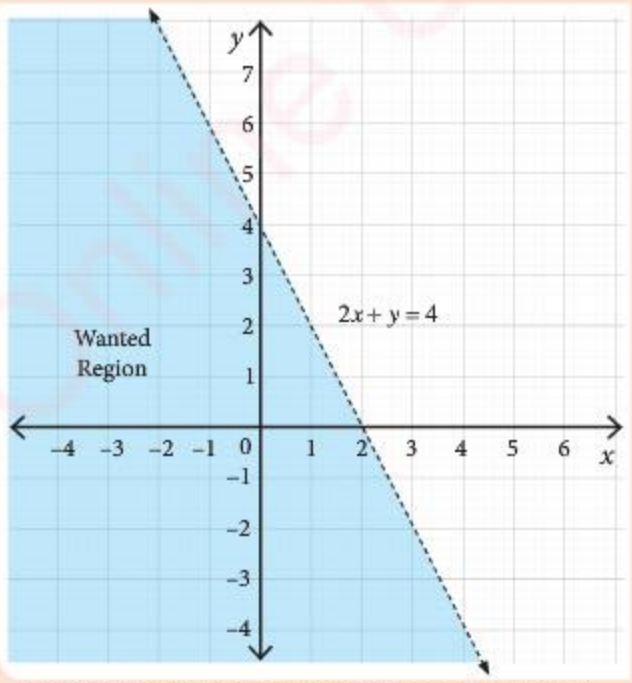

Figure 11.1 Graphical representation of the inequality $2x + y < 4$.

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Choose any convenient test point which does not lie on the boundary line. The origin (0, 0) is a good choice for easy testing. Substitute $x=0$ and $y=0$ into the inequality

i.e. $2(0)+0<4.$

$0+0<4.$

$0<4$ a true statement

Because the inequality is true for the test point, then it is a wanted region. Shade the wanted region as shown in the Figure 11.1.

Example 11.4: Draw the graph of $3x+2y\geq 6$

Solution

First, replace the inequality sign with an equal sign to form an equation of the boundary line of the graph; i.e. $3x+2y=6$. Then, draw the graph by using the equation given, i.e. $3x+2y=6$ Table 11.6 shows x and y intercepts:

Table 11.6 The x and y intercepts for $3x+2y=6$

x	0	2
y	3	0

Select a point (not on the line), say (0, 0), which lies on the plane (see Figure. 11.2) and determine if this point satisfies the given inequality. That is; $3(0)+2(0)\geq 6$ $0\geq 6$ a false statement.

Hence, region **I** is the wanted region of the given inequality. Now, shade region **I** which is the wanted region. Figure 11.2 represent the graph of the inequality $3x+2y\geq 6$

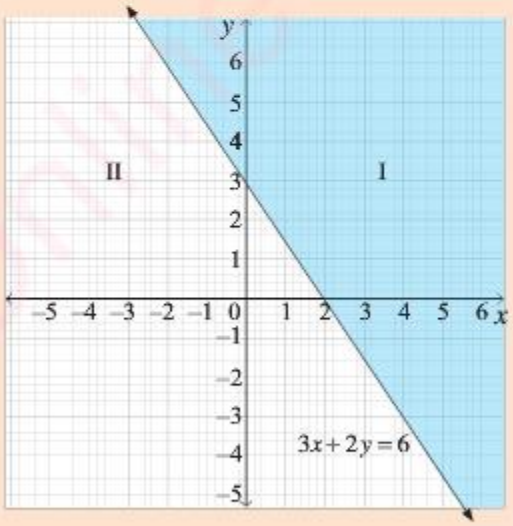

Figure 11.2 Graphical representation of the inequality $3x+2y\geq 6$

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Example 11.5: Graph the following set of inequalities in the same xy plane. Shade the wanted region.

$$2x + y \leq 10$$
$$2x + 3y \leq 18$$
$$x \geq 0,\ y \geq 0$$

Solution

First, replace the inequality sign with an equal sign to form equations of the boundary lines of the graph:

i.e. $2x + y = 10$
$2x + 3y = 18$
$x = 0$ and $y = 0$

Draw the graph using the given equations. The values of x and y-intercepts for equations $2x + y = 10$ and $2x + 3y = 18$ are shown in Tables 11.7 and 11.8 respectively.

Table 11.7 The values of the x and $y-$ intercepts for $2x + y = 10$

x	0	5
y	10	0

Table 11.8 The values of the x and $y-$ intercepts for $2x + 3y = 18$

x	0	9
y	6	0

Its graph is as shown in Figure 11.3

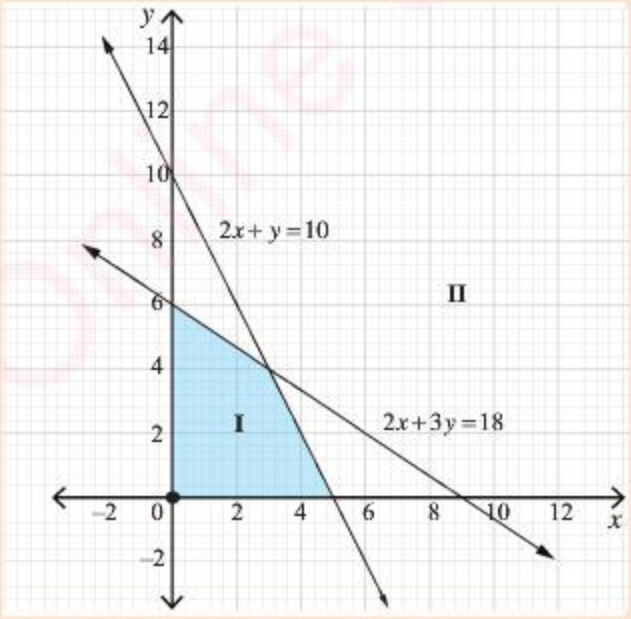

Figure 11.3 Graphical representation of the set of inequalities

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Choose any convenient test point, not on the boundary lines. The origin (0, 0) is a good choice because it makes the calculations easier (see Figure. 11.3)

Substitute $x=0$ and $y=0$ in the inequality $2x+y\le 10$ to test if the statement is true or false. That is;

$$2x+y\le 10$$
$$2(0)+(0)\le 10$$
$$2(0)+0\le 10$$
$$0\le 10,\ \text{a true statement}$$

Hence, plane I is the wanted region of the given inequality ($2x+y\le 10$).

Substitute $x=0$ and $y=0$ in the inequality $2x+3y=18$ to test if the statement is true or false. That is;

$$2x+3y\le 18$$
$$2(0)+3(0)\le 18$$
$$0+0\le 18$$
$$0\le 18,\ \text{a true statement}$$

Hence, plane I is the region of the given inequality $2x+3y\le 18$. Now, shade plane I which is the wanted region (see Figure 11.3). For $x\ge 0$ and $y\ge 0$, shade the positive sides to avoid negativity.

Exercise 11.2

1. Draw separate graphs for each of the following and show the regions representing the inequalities:
   a) $y<2x-1$
   b) $5x\ge 2y+10$
2. Draw a graph of the set of points whose coordinates satisfy the following inequalities:
   $5x+6y\le 60,\ x\le 6,\ x\ge 0,$ and $y\ge 0$
3. Show by shading the wanted regions of the planes representing each of the following simultaneous inequalities:
   a) $y\le 2x-1,\ y\ge 3-x,\ x\le 4$
   b) $5x+y\ge 15,\ x+2y\ge 12,$ $x\ge 0,$ and $y\ge 0$
   c) $2x+y\ge 10,$ $4x+y\ge 12,$ $x+5y\ge 15,$ $x\ge 0,$ and $y\ge 0$
4. Figure 11.4 shows the boundary lines for the set of inequalities $x\le 7,\ x+y\ge 6$ and $y\le x$. Shade the wanted region.

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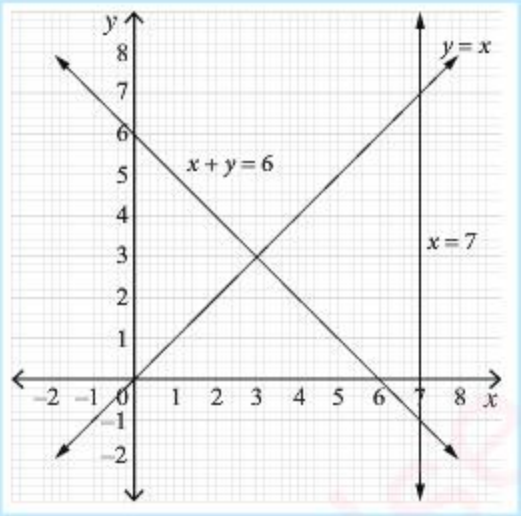

Figure 11.4 Graphical representation of the linear enequalities $x \leq 7$, $x + y \geq 6$ and $y \leq x$

5. The boundary line of the inequality $2x + y > 8$ is shown in Figure 11.5. Shade the wanted region.

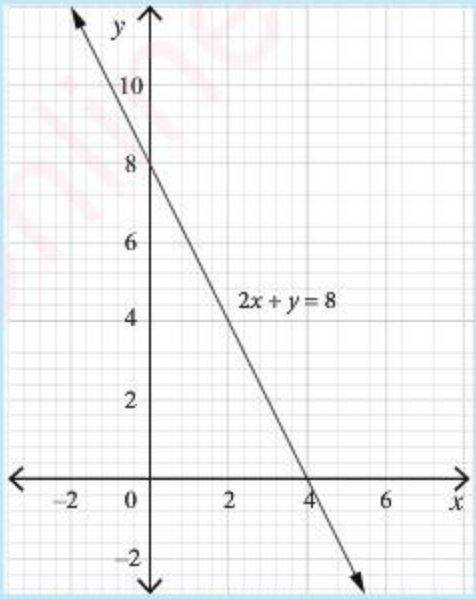

Figure 11.5 Graphical representation of the linear inequality $2x + y > 8$

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11.4 Solutions of linear programming problems

In solving linear programming problems, among the methods used is the graphical representation of the inequalities. This method is used when there are only two decision variables.

The following are important terminologies that are connected with graphical method of solving linear programming problems.

1. ***Solution:*** Is a set of values of decision variables satisfying all the constraints of a linear programming problem.
2. ***Feasible region:*** Is the common region satisfying all the constraints including the non-negativity restriction of a linear programming problem. Feasible regions can be bounded (i.e. completely enclosed) or unbounded (i.e. have infinite set of solutions)

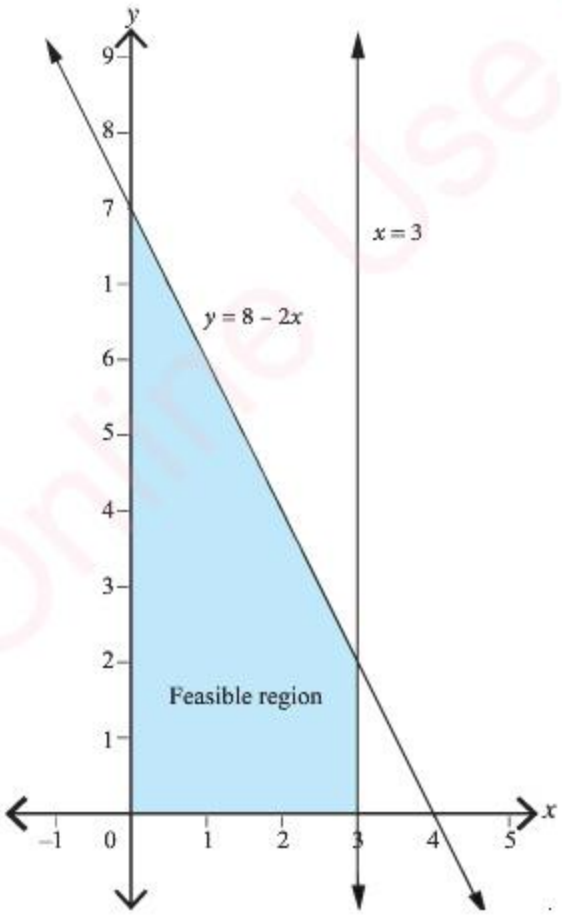

Figure 11.6 Bounded feasible region

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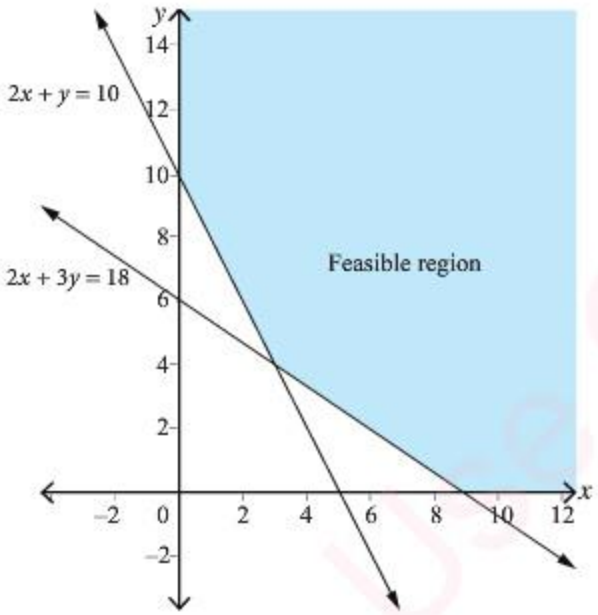

Figure 11.7 Unbounded feasible region

3. ***Corner points:*** These are vertices of the feasible region. The corner points are obtained by either inspecting the coordinates of the corners of a well-drawn feasible region or by solving for points of intersection of the boundary lines.
4. ***Optimal solution:*** Is any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function. If there is an optimal solution to a linear programming problem, then the optimal solution will occur at one or more corner points.
5. ***Optimal value:*** Is a maximum or minimum value of an objective function over the feasible region of an optimization problem.

The following are outlined steps for solving linear programming problems by graphical method.

a) Make a summary in tabular form showing all the given information.

b) Identify the decision variables (unknowns) by letters.

c) Formulate the objective function from the linear programming problem.

d) Write the set of constraints (inequalities) according to the given conditions.

e) Draw the graph using the constraints to determine the feasible region.

f) Identify the corner points of the feasible region.

g) By testing the corner points in the objective function, determine the optimal solution and optimal point.

h) Make the conclusion basing on the requirements of the problem.

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Example 11.6: Clement has Tsh 7,200 to spend on buying counterbooks. At the school shop a counterbook is sold at Tsh 480 and at the stationery store it costs Tsh 720. The school shop has only 12 counter books left and Clement wants to obtain the greatest number of counter books possible using the money he has. Find the greatest number of counter books that Clement can buy.

Solution

Table 11.9 Summary of the places of buying counterbooks and its respective cost

Place	Cost in Tsh	Available
Shop	480	12
Stationery	720	
Total available cash	7200	

Let x be the number of counter books bought at the school shop.

Let y be the number of counter books bought at the stationery store.

Objective function: maximize:

$f(x, y) = x + y$

Constraints:

$480x + 720y \leq 7200 \Rightarrow 2x + 3y \leq 30$

$x \leq 12$

$x \geq 0$

$y \geq 0$

Table 11.10 The values of the x and y intercepts for the inequation $2x + 3y = 30$

x	0	15
y	10	0

Figure 11.8 represents the graph of the the linear programming problem.

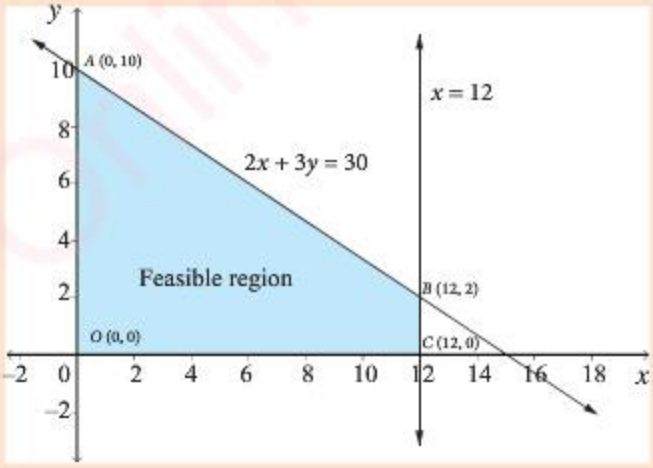

Figure 11.8 Graphical representation of feasible region

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Testing corner points:

Table 11.11 Testing corner points using the objective function

Corner points	Maximize $f(x, y) = x + y$
A(0, 10)	10
B(12, 2)	14
C(12, 0)	12
O(0, 0)	0

Therefore, the greatest number of counter books that Clement can buy is 14; i.e. 12 counter books from the school shop and 2 counter books from the stationery store.

Example 11.7: A person must fulfil certain minimum daily requirements for several kinds of nutrients. Assume that there are only three types of nutrients; vitamin, protein and fat. The person's diet consists of only two food items, F_1 and F_2 whose cost and nutrient contents are shown in the Table 11.12.

Table 11.12 Type of nutrients in foods F_1 and F_2 and their cost

	NUTRIENTS			
FOOD	VITAMIN	PROTEIN	FAT	COST(Tsh)
F_1	20	40	30	5000
F_2	50	20	40	3000
Minimum daily requirement	800	1000	1200	

a) Formulate a linear programming problem.

b) What combination of two food items will satisfy the daily requirement and entail the least cost?

Solution

Let x be the number of units of food F_1 and y be the number of units of food F_2

a) The objective is to Minimize

$f(x, y) = 5000x + 3000y$

Subject to the constraints:

$20x + 50y \geq 800 \Rightarrow 2x + 5y \geq 80$

$40x + 20y \geq 1000 \Rightarrow 2x + y \geq 50$

$30x + 40y \geq 1200 \Rightarrow 3x + 4y \geq 120$

$x \geq 0$

$y \geq 0$

b) The x and y intercepts for the equations $2x + 5y = 80$, $3x + 4y = 120$ and $2x + y = 50$ are shown in Figures 11.13, 11.4 and 11.15, respectively.

Table 11.13 The x and y intercepts for $2x + 5y = 80$

x	0	40
y	16	0

Table 11.14 The x and y intercepts for $3x + 4y = 120$

x	0	40
y	30	0

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Table 11.15 The x and y intercepts for $2x + y = 50$

x	0	25
y	50	0

Figure 11.9 represent the graph of the linear programming problem

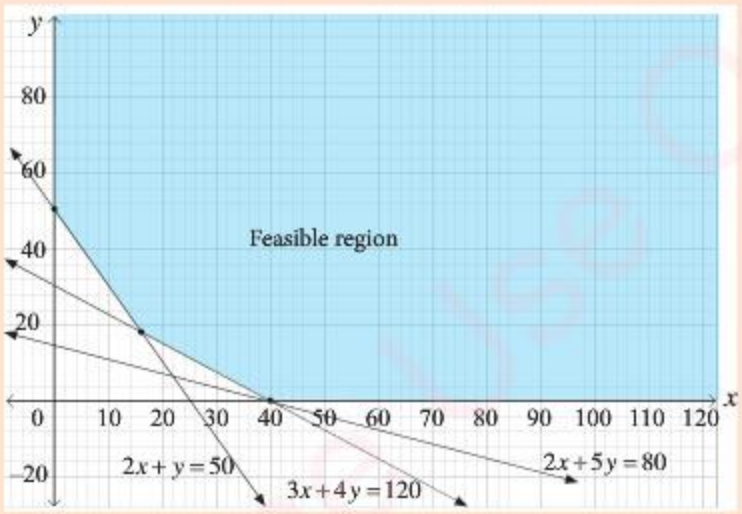

Figure 11.9 Graphical representation of the feasible region

Table 11.16 Testing corner points using objective function

Corner points	$f(x, y) = 5000x + 3000y$ (Tsh)
A(0,50)	150,000
B(16,18)	134,000
C(40,0)	200,000

In order to entail the least cost of Tsh 134,000, 16 units of food F_1 and 18 units of food F_2 should be produced to satisfy the daily requirements.

Example 11.8: A firm makes two types of curtains, ordinary and deluxe. Each ordinary curtain takes 1.5 hours (1 hour and 30 minutes) to produce and a deluxe takes 3 hours to produce and use 12 metres and 14 metres of materials, respectively. Workers of the firm can work for a total of 30 hours and there is 180 metres of material available. If the profit for ordinary and deluxe curtains is Tsh 18,500 and Tsh 25,500, respectively;

a) Formulate a linear programming problem.

b) How many of each should be made

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to maximize the profit?

Solution

Table 11.17 Summary of the types of curtains, time and materials of making each type of curtain

Type of curtain	Time (hours)	Material (metre)	Profit (Tsh)
Ordinary	1.5	12	18,500
Deluxe	3	14	25,500
Total available	30	180	

Let x be the number of ordinary curtains and y be the number of deluxe curtains.

a) The linear programming problem is:

Objective function: maximize profit:
$f(x, y) = 18500x + 25500y$

Subjected to constraints:

$1.5x + 3y \le 30 \Rightarrow x + 2y \le 20$

$12x + 14y \le 180 \Rightarrow 6x + 7y \le 90$

$x \ge 0$

$y \ge 0$

b) Tables for x and y intercepts

Table11.18 The x and y intercepts for $x + 2y = 20$

x	0	20
y	10	0

Table 11.19 The x and y intercepts for $6x + 7y = 90$

x	0	15
y	12.9	0

The graph of the linear programming problem is presented in Figure 11.10.

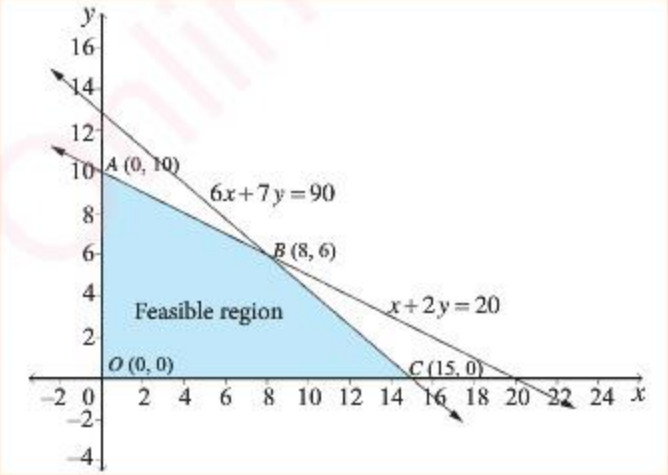

Figure 11.10 Graphical representation of feasible region for the maximazation problem

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Table 11.20 Testing corner points using the objective function

Corner point	$f(x, y) = 18500x + 25500y$ (Tsh)
O(0,0)	Tshs. 0
A(0,10)	255,000
B(8,6)	301,000
C(15,0)	277,500

Hence, the maximum profit is Tsh 301,000. Therefore, the firm should make 8 ordinary curtains and 6 deluxe curtains so as to get the maximum profit.

Example 11.9: The manufacturing company has two plants, P and Q, which produce three types of products namely L, M and N. The first plant can manufacture three units of L, twelve units of M and twelve units of N at the cost of Tsh 24,000 per hour. The second plant can manufacture nine units of L, fifteen units of M and three unit of N at the cost of Tsh 18,000 per hour. The company has received an order of 204 units of L, 690 units of M and 330 units of N. How many hours should be given to each plant in order to satisfy the order at the least cost?

Solution

Table 11.21 Products from Plants P and Q and its cost of production

PLANTS	PRODUCTS			COSTS(Tsh)
	L	M	N	
P	3	12	12	24,000
Q	9	15	3	18,000
Number of units required	204	690	330	

Let x represent the number of hours of plant P and y represent the number of hours of plant Q

Minimize: $f(x, y) = 24000x + 18000y$

Subject to:
$$3x + 9y \geq 204 \Rightarrow x + 3y \geq 68$$
$$12x + 15y \geq 690 \Rightarrow 4x + 5y \geq 230$$
$$12x + 3y \geq 330 \Rightarrow 4x + y \geq 110$$
$$x \geq 0$$
$$y \geq 0$$

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Tables 11.22, 11.23 and 11.24 represent x and y intercepts

Table 11.22 The x and y intercepts for $x+3y=68$

x	0	68
y	22.7	0

Table 11.23 The x and y intercepts for $4x+5y=230$

x	0	57.5
y	46	0

Table 11.24 The x and y intercepts for $4x+y=110$

x	0	27.5
y	110	0

Figure 11.11 represent the graph of the linear programming problem

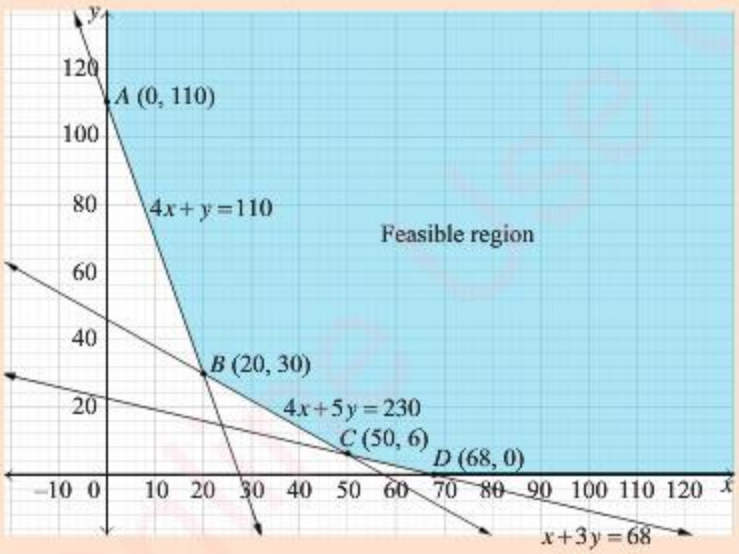

Figure 11.11 Graphical representation of feasible region for the minimization problem

Table 11.25 Testing corner points using the objective function

Corner points	$f(x, y) = 24000x + 18000y$ (Tsh)
A (0, 110)	1,980,000
B (20, 30)	1,020,000
C (50, 6)	1,308,000
D (68, 0)	1,632,000

The least cost is Tsh 1,020,000. Therefore, the company should give 20 hours and 30 hours to P and Q, respectively in order to satisfy the order at the least cost.

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Example 11.10: A retail shop received orders from customers C_1 and C_2 for the following food packages. The package for C_1 should contain 50 kg of beans, 30 kg of rice, and 70 kg of maize while the package for C_2 should contain only 25 kg of beans and 45kg of rice. The stock has 850 kg of beans, 810 kg of rice and 980 kg of maize. If a unit of package C_1 costs Tsh 2,100 and that of C_2 costs Tsh 1,800;

a) formulate a linear programming problem.

b) how many packages should be supplied by the retailer so as to realize a maximum sale?

c) how much stock remains after meeting the orders?

Solution

Table 11.26 Summary of the food parckages and respective costs for customers C_1 and C_2

Customers	Packages (in kg)			
	Beans in kg	Rice in kg	Maize in kg	Cost per package in Tsh.
C_1	50	30	70	2100
C_2	25	45	0	1800
Stock	850	810	980	

Let x be the number of packages for C_1 and y be the number of packages for C_2

a) The linear programming problem is:

Objective function:maximize: $f(x, y) = 2100x + 1800y$

Subject to:

$50x + 25y \le 850 \Rightarrow 2x + y \le 34$

$30x + 45y \le 810 \Rightarrow 2x + 3y \le 54$

$70x \le 980 \Rightarrow x \le 14$, $x \ge 0$, $y \ge 0$

b) Tables 11.27 and 11.28 represent x and y intercepts

Table 11.27 The x and y intercepts for $2x + y = 34$

x	0	17
y	34	0

Table 11.28 The x and y intercepts for $2x + 3y = 54$

x	0	27
y	18	0

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The graph of the linear programming problem is presented in Figure 11.12.

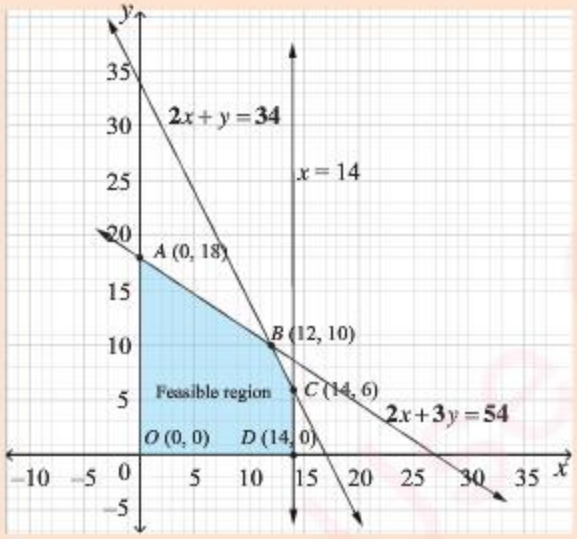

Figure 11.12 Graphical representation of feasible region of maximization problem

Table 11.29 Testing corner points using the objective function

Corner points	$f(x, y) = 2100x + 1800y$ (Tsh)
O(0,0)	0
A(0,18)	32,400
B(12,10)	43,200
C(14,6)	40,200
D(14,0)	29,400

To maximize sales, 12 packages for customer C_1 and 10 packages for customer C_2 should be supplied.

c) 12 packages for C_1 and 10 packages for C_2 , amount supplied is as follows:

Beans: $(12\times50)+(10\times25)=850\,kg$: All supplied

Rice: $(12\times30)+(10\times45)=810\,kg$: All supplied

Maize: $(12\times70)+0=840\,kg$ but amount of maize available = $980\,kg$

$\therefore 140\,kg$ of maize remains after meeting the orders.

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Exercise 11.3

1. Maximize: $f(x, y) = 10x + 20y$
   Subject to: $x + 3y \leq 12$
   $x + y \leq 8$
   $x \geq 0,\ y \geq 0$

2. Minimize: $f(x, y) = 4x + 5y$
   Subject to: $x + y \geq 10$
   $2x + y \geq 12$
   $6x + y \geq 16$
   $x \geq 0,\ y \geq 0$

3. Minimize the objective function $f(x, y) = 30x + 50y$ subject to the following constraints:
   $6x + y \geq 18,\ x + 4y \geq 12,$
   $2x + y \geq 10,\ x \geq 0$ and $y \geq 0$

4. James is going to sow the maize and beans. He estimates to use 4 men and 6 men per hectare of maize and beans, respectively. He has 26 men available. Maize costs Tsh 1,200 per hectare and beans costs Tsh 800 per hectare and he is prepared to spend up Tsh 4,800. Find the greatest possible area of the land James can sow.

5. In a Biology examination consisting of Paper One (P_1) and Paper Two (P_2), both papers are marked out of 100. A candidate score is represented by marks x for P_1 and marks y for P_2. The candidate will be considered to have passed if the function $5x + 8y$ is at least 400 but candidates must score at least 24 marks on P_1 and 30 marks on P_2. Find the lowest value of $x + y$ for any candidate who passes and give the corresponding values of x and y.

6. Amina wants to use her farm to plant maize and millet. The farming of maize costs Tsh 12,000 per acre and millet costs Tsh 20,000 per acre. The funds available in farming the two crops is Tsh 240,000 and there are only 13 acres available. Maize requires 10 man hours per acre while millet require 8 man hours per acre and there are 120 man hours available. The estimated profit is Tsh 18,000 per acre of maize and Tsh 15,000 per acre of millet.
   a) How should she allocate her farm for maximum profit?
   b) What is the maximum profit?

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Revision exercise 11

1. Find the corner points for the system of linear inequalities below:

   $x+y\leq 4,\ x-y\leq 3,\ x\leq 3,\ x\geq 0$ and $y\geq 0$

2. Explain the following terms:
   a) Linear programming
   b) Optimal solution
   c) Optimal value

3. Draw the graph of the following inequalities showing the feasible region and the corner points and then determine the maximum and minimum values of the function $f(x, y)=3x+5y$ and

   constraints $\begin{cases} x+y\leq 12 \\ x\geq 3 \\ y\geq 2 \end{cases}$

4. A calculator company produces a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 100 scientific and 80 graphing calculators each day. Due to some limitations on the production capacity, no more than 200 scientific and 170 graphing calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators must be shipped each day. If each scientific calculator sold results in Tsh 20,000 loss, but each graphing calculator produces Tsh 50,000 profit, how many of each type should be made daily to maximize the net profits?

5. An animal meal is to be made from two different kinds of food. Each bag of food I contains 3 units of nutrients N_1 and N_2, and 4 units of nutrients N_3 and it costs Tsh 8,000. Each bag of food II contains 1, 3, and 4 units of nutrients N_1, N_2, and N_3, respectively and it costs Tsh 6,000. For nutritional purposes, each meal is required to contain at least 21 units of N_1, 30 units of N_2, and 60 units of N_3. How many bags of each type of food should be ordered to satisfy the nutritional requirements at a minimum cost?

6. Suppose that 8, 12, and 9 units of proteins, carbohydrates, and fats, respectively are the minimum weekly requirement for the diet of a person. Food A contains 2, 6, and 1 units of proteins, carbohydrates, and fats respectively per kilogram; and food B contains 1, 1, and 3 units of proteins, carbohydrates, and fats, respectively per kilogram. If food A costs Tsh 500 per kilogram and food B costs Tsh 400 per kilogram, how many kilograms of each type of food should a person buy per week to minimize the cost and still meet his requirement?

7. Ashura needs to buy some filing cabinets. She knows that cabinet C_1 costs Tsh 100,000 per unit, requires six square centimetre of floor space, and holds eight cubic centimetres of files. Cabinet C_2 costs Tsh 200,000 per unit, requires eight square centimetres of floor space, and holds twelve cubic centimetres of files. Ashura has been given Tsh 1,400,000 for this purchase. The office has a room for no more than 72 square centimetres of cabinets. How many of each model should she buy in order to maximize storage volume?

8. A manufacturer has 24, 36, and 18 tons of wood, plastics, and steel, respectively. To make a possible product A, it requires 1, 3, and 2 tons of wood, plastic, and steel, respectively; while the product B requires 3, 3, and

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1 tons of wood, plastic, and steel, respectively. Product A sells for Tsh 400,000 and B sells for Tsh 600,000. How many products of each should the manufacturer make to obtain the maximum gross income?

9. A factory manufactures two types of machines A and B. Each type is made of certain type of alloy metal. The factory has only 520 kg of this metal in a day. 20 kg is required to manufacture machine A and 50 kg is required for machine B. Each machine requires 20 minutes assembling time and the assembling department has only 280 minutes. Furthermore, machines A and B require 10 minutes and 5 minutes, respectively to be painted. The painting department is restricted to use not more than 200 minutes in a day. The company earns the profit of Tsh 25,000 and Tsh 28,000 for type A and B, respectively. How many of each should be produced for maximum profit? What is the maximum profit?

10. The wood work department makes tables and chairs which are processed through assembling and finishing section. For a particular college, 48 hours are available in assembling and 36 hours in finishing sections. To make one table, it requires 3 hours in assembling and 3 hours in finishing sections, while a chair requires 4 hours in assembling and 2 hours in finishing sections. If a chair fetches Tsh 400 as profit and a table fetches Tsh 500, how many chairs and tables can be made to maximize the profit?

11. Two friends are planning a fitness exercise programme. They can spend up to 1.5 hours per day on exercise. Some of the time will be spent on aerobics and the remainder on flexibility exercise. They would prefer to do more aerobics than flexibility, but could not manage more than 60 minutes of aerobics. They discovered that aerobics use 8 calories per minute and flexibility exercises use 3 calories per minute. They wish to lose the maximum number of calories. At which point will they find this maximum loss of calories and what is the maximum calories loss?

12. A farmer has decided to plant coffee and maize on his 20 acres farm. He has Tsh 220,000 capital available and he can work for 30 days in a year, i.e. 2 days per acre in a coffee farm and 3 days per acre in a maize farm. The cultivation cost is Tsh 20,000 per acre of coffee and Tsh 10,000 per acre of maize. If the profit is Tsh 180,000 and Tsh 100,000 per acre of coffee and maize, respectively, how many acres a farmer has to plant in order to maximize the profit?

13. A farmer intends to purchase fertilizer containing three nutrients A, B, and C. The minimum requirements are 160, 200, and 80 units of A, B, and C, respectively. There are two popular brands of fertilizers in the market: brand I and brand II. Brand I costs Tsh 4,000 per bag and contains 3, 5, and 1 unit of A, B, and C, respectively. Brand II costs Tsh 3,000 per bag and contains 2 units of each nutrient. If the farmer wishes to minimize the total cost while maintaining the requirements;

    a) how many bags of each brand should the farmer buy?

    b) what is the minimum total cost?

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Answers to the exercises

CHAPTER ONE

Exercise 1.1

1. a) $2{\cdot}43\times10^{5}$
   b) $5{\cdot}46\times10^{0}$
   c) -4.27×10^{1}
2. a) 8.12×10^{-2}
   b) 7.7×10^{-2}
3. $1.5004, (\text{in 4 d.p})$
4. a) $0.4167, (\text{in 4 d.p})$
   b) 268
5. a) 160.9344 km
   b) 100 °C
   c) 3280.8399 ft
6. $\bar{x} = 67.62$
   $\delta_x = 1.2472$
   $\sum x^2 = 228701$
7. a) (i) $x = -502,\ y = -182$
   (ii) $x = 13,\ y = -2.1667,\ z = -1.6667$
   b) $p = 2.5,\ p = -4$ or $p = 0.6667$
8. a) $\begin{pmatrix} -7.7857 & -0.5714 & 1.1429 \\ 23.7857 & 2.0714 & -3.6429 \\ -13.9286 & -1.3571 & 2.2143 \end{pmatrix}$
   b) $\begin{pmatrix} -9 & -13 & -14 \\ -58 & -81 & -89 \\ -99 & -150 & -156 \end{pmatrix}$
   c) -28

Exercise 1.2

1. a) 7.845265
   b) 471.851312
2. a) 2.234828
   b) -0.662497
3. $\bar{x} = 39.692308$
   $\sum x^2 = 30370$
   $\delta_n = 27.580329$
4. a) 1.047198
   b) $404685.6\,\text{m}^2$
   c) $0.001\ \text{m}^3$
   d) 1litre
5. a) 4 months and 3 days
   b) 31 years, 3 months, 3 weeks and 2 days
   c) 17 years, 11 months, 3 weeks and 4 days

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Exercise 1.3

1. a) Letters P, Q, R and S represent cells in the excel spreadsheet package

   b) Letter P is naming cell A_3

   Letter Q is naming cell F_1

   Letter R is naming cell C_6

   Letter S is naming cell D_4

3. a) −20.4 b) 425.5 c) 9.95548 d) 1

6. Mean = 49.6111

   Median = 53

   Mode = 73

   $\text{var}(x) = 718.12654$

   $\delta_x = 26.79788$

Revision Exercise 1

1. a) 11.35
   b) 58.2195
   c) 1,751.258622

2. a) −3.1793
   b) 1.0243

3. a) 17.3673
   b) 30.83333
   c) 1310

4. a) $(x, y) = (30, 50)$
   b) $(x, y) = (60, 90)$

5. a) $(x, y, z) = (1, 3, 3)$
   b) $(x, y, z) = (3, -2, 1)$

6. a) $x_1 = 0.8333,\ x_2 = 0.75$
   b) $x_1 = 16,\ x_2 = -25$
   c) $x_1 = 2,\ x_2 = -9,\ x_3 = 12$
   d) $x_1 = 1.5,\ x_2 = -0.4,\ x_3 = 0.3333$

7. a) 62.1
   b) 26.3522
   c) 12,420
   d) 910,170

10. a) 53.45
    b) 536.83
    c) 23.17
    d) 50,911.36
    e) 801.8

11. a) −0.0949
    b) 4330
    c) 75,750.51
    d) 1,772,776,810

12. a) 0.8858
    b) 0.9917
    c) $\bar{x} = 161$ and $\sigma = 19.51$

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13. a) $\begin{pmatrix} 106 & 90 & -46 \\ 11 & 9 & 27 \\ 50 & -72 & 14 \end{pmatrix}$

b) $\begin{pmatrix} 0.0587 & 0.3929 & 0.1034 \\ 0.0771 & -0.2263 & 0.1930 \\ 0.1870 & 0.0624 & -0.0567 \end{pmatrix}$

c) $\begin{pmatrix} 106 & 11 & 50 \\ 90 & 9 & -72 \\ -46 & 27 & 14 \end{pmatrix}$

d) 384,192

14. a) 58.85

b) 26.1472

c) 683.6775

d) 829,400

15. a) 176.67 °C

b) 8,202.1 feet

c) 643.7376 km

d) 576 km/h

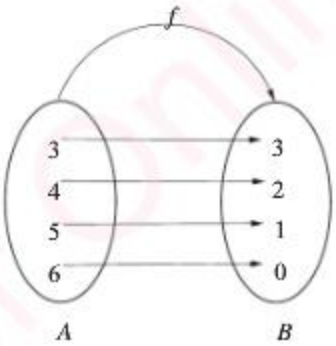
e) 10,319,482.8 m²

CHAPTER TWO

Exercise 2.1

1. a) (i) Domain = $\{3, 4, 5, 6\}$ and Range = $\{0, 1, 2, 3\}$

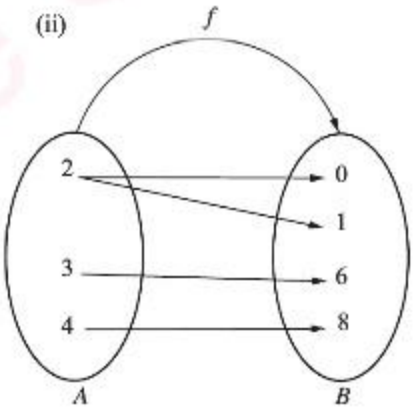
(ii)



(iii) It is a function

b) (i) Domain = $\{2, 3, 4\}$ and Range = $\{0, 1, 6, 8\}$

(ii)



(iii) It is not a function

(c) (i) Domain = $\{1, 2, 3\}$

Range = $\{1, 2, 6\}$

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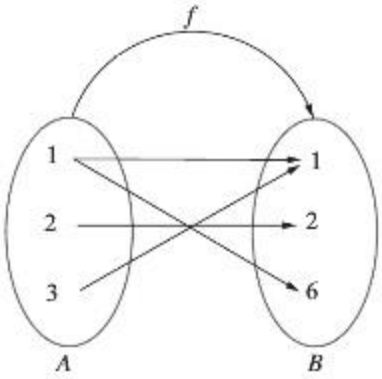
(ii)



(iii) It is not a function.

2. (a) Domain = $\{-1, 0, 2, 3\}$ and Range = $\{-1, 0, 8, 27\}$

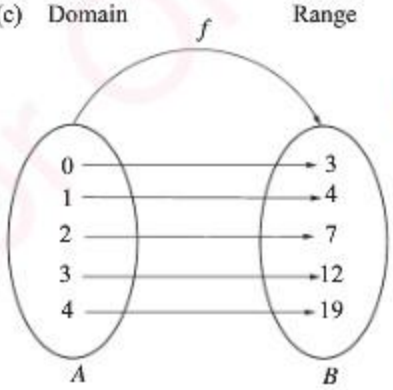
(b) Set of ordered pairs

$\{(-1,-1), (0,0), (2,8), (3,27)\}$

3. (a) Range = $\{3,4,7,12,19\}$ (b) set of ordered pair

$\{(0,3),(1,4),(2,7),(3,12),(4,19)\}$

(c) Domain Range



4. a) It is a function

Reason: mapping is many to one.

b) It is a function

Reason: Mapping is many to one

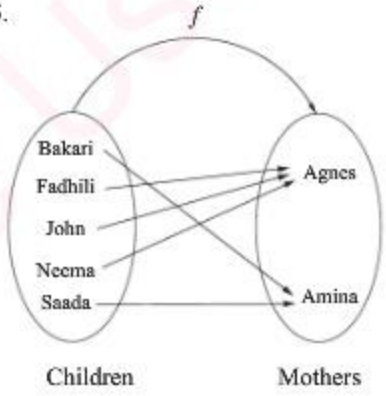
c) It is not a function

Reason: A mapping of one to many

d) It is a function

Reason: It is a mapping of one to one

5.



Exercise 2.2

1. (a) x^2+9x+2 (b) $4x^3+6x^2+8x+12$

2. (a) $3x-10$ (b) $\dfrac{6}{x+2}$

3. (a) $4x^3-2x^2+6x+18$, cubic polynomial of degree 3

(b) $\dfrac{9x-6}{2x^2-x-1}$ is not polynomial

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4. (a) $-2x-2$, linear polynomial, degree 1

(b) $-5x^3-4x-6$, cubic polynomial, degree 3.

Exercise 2.3

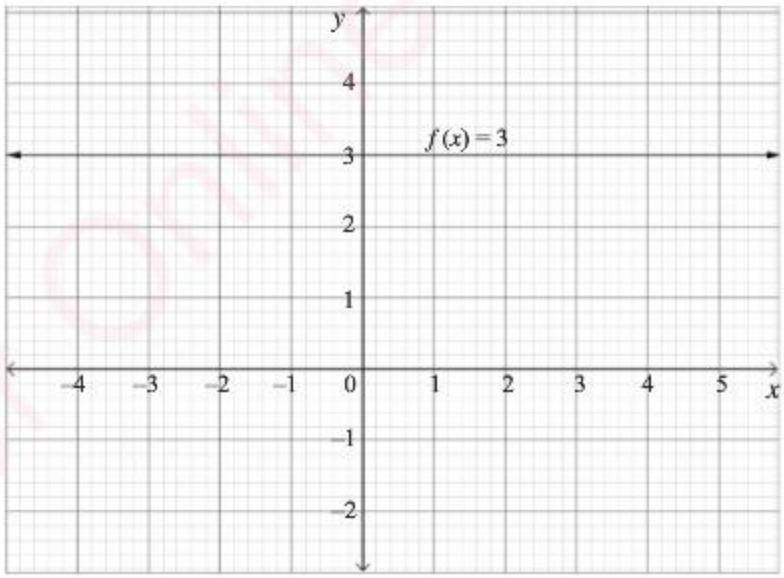
1. $2x^3+11x^2+10x-3$
2. x^3+4x^2+x-6
3. $6t^3-5t^2-3t+2$
4. $\dfrac{10}{x^2+6x+9}$
5. $x^5-2x^4+x^3-8x^2+16x-8$, polynomial of degree 5.

Exercise 2.4

1. $3+\dfrac{4}{x-1}$
2. $x+1$
3. x^3+6x^2-2x+3
4. $2x^2+9x+8+\dfrac{11}{x-1}$ and remainder is 11.
5. x^2-1; factors are $(x-3)$, $(x-1)$ and $(x+1)$

Exercise 2.5

1.



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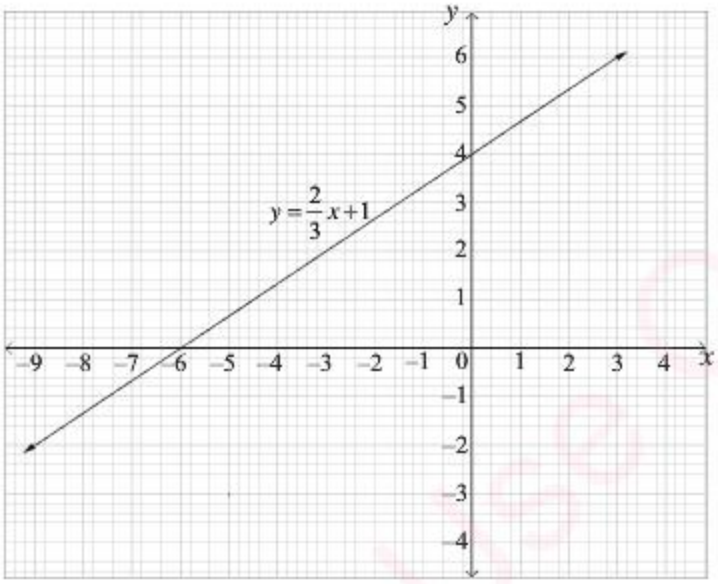
2.



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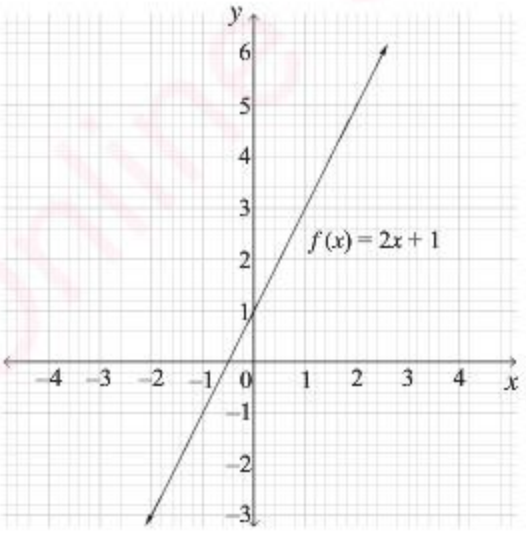

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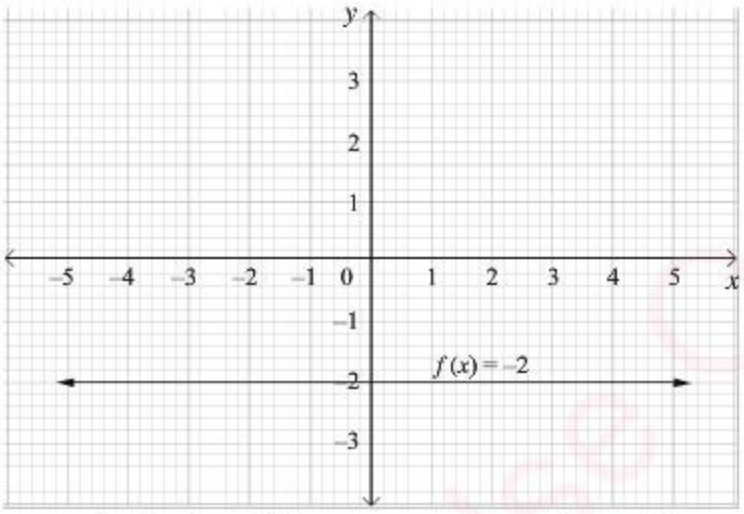
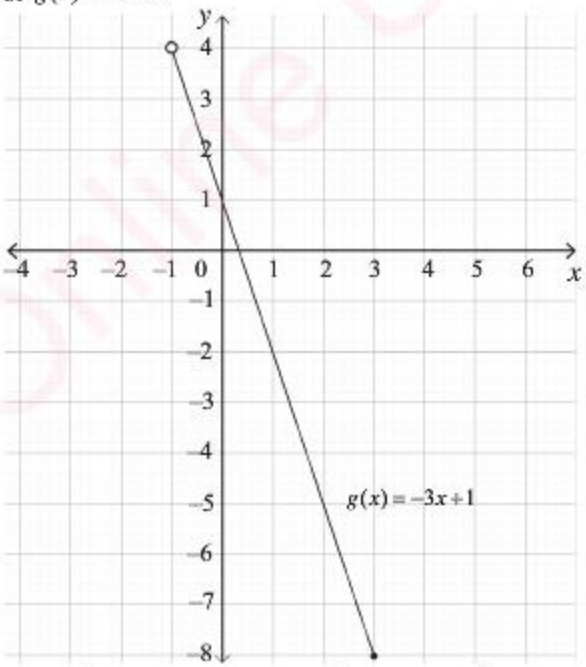
4.



Domain $= \{x : x \in \mathbb{R}\}$ Range $= \{y : y = -2\}$

5. The graph of $g(x) = -3x + 1$



Domain $= \{x : -1 < x \leq 3$, Range $= \{y : -8 \leq y < 4\}$

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Exercise 2.6

1. a) (3,5)

b) Minimum

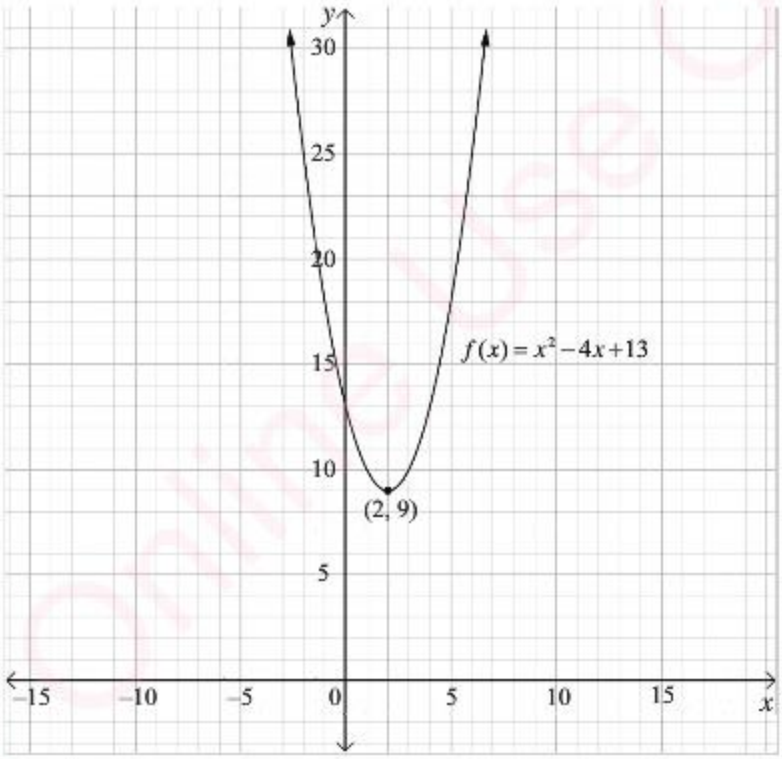
c) $x=3$

d) Domain $=\{x: x\in\mathbb{R}\}$ and Range $=\{y: y\geq 5\}$

2. a) (2,9), Minimum

b) Domain $=\{x: x\in\mathbb{R}\}$, Range $=\{y: y\geq 9\}$

c) The graph of $f(x)=x^2-4x+13$ is shown below



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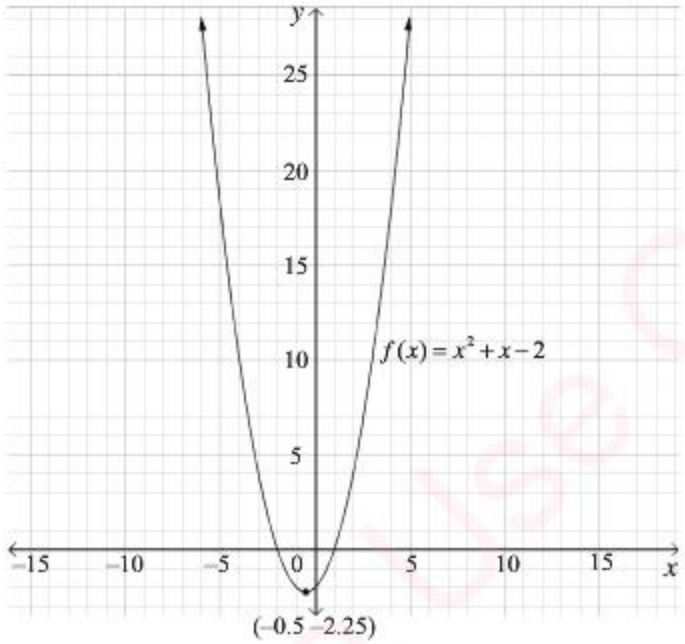
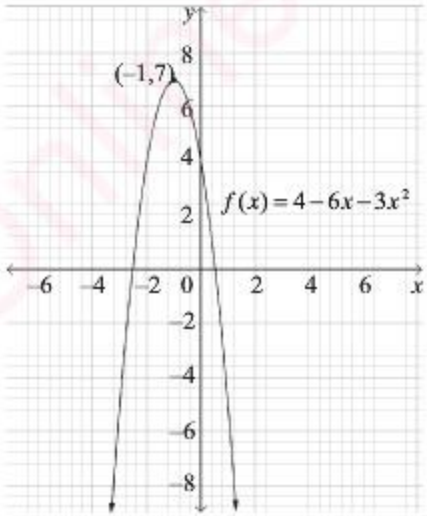
3 .a)



b)



4. (3,15), Maximum, $x=3$
5. a) 5.1 sec or 0.5 sec
   b) 41.2 metres

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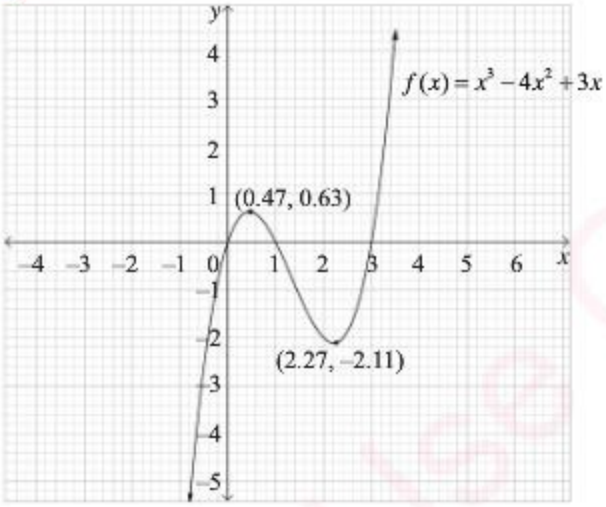
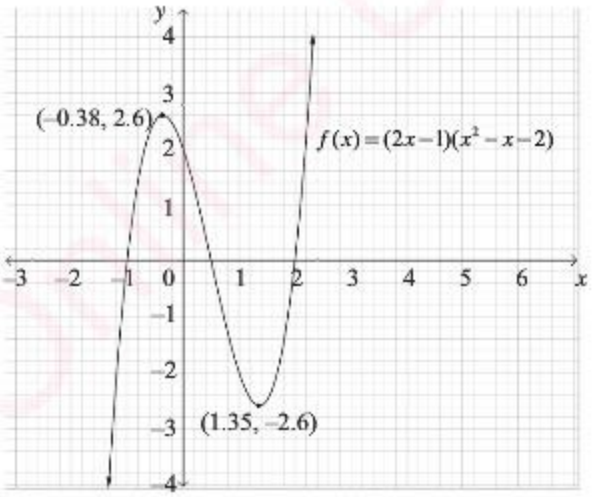
Exercise 2.7

1.



2.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

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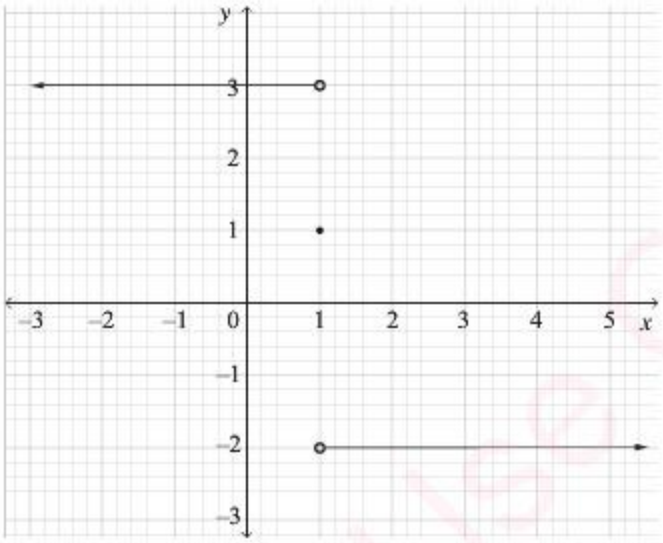
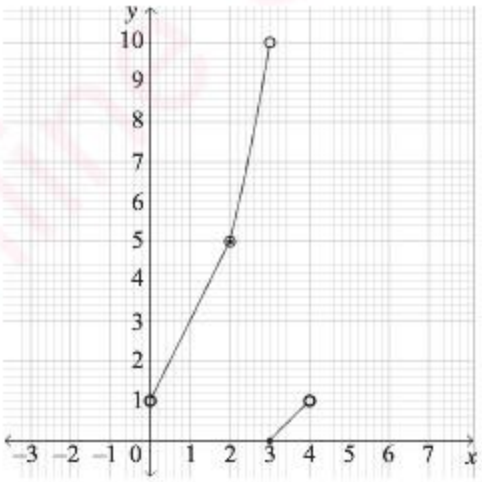
3.



4.



Domain $= \{x : 0 < x < 4\}$

Range $= \{y : 0 \leq y < 10\}$

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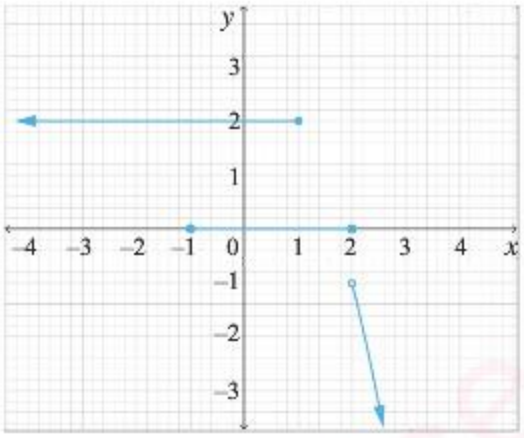

a) Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y < -1,\ y = 0,\ y = 2\}$

b) (i) $f(-1) = 0$ and 2

(ii) $f(2) = 0$

(iii) $f(3) = -6$

Exercise 2.8

1. a) Horizontal asymptote, $y = 0$ and vertical asymptote, $x = -1$

Domain $= \{\in \mathbb{R} : x \neq -1\}$ and range $= \{\in \mathbb{R} : y \neq 0\}$

b) Horizontal asymptote, $y = 0$ and vertical asymptote, $x = \frac{3}{4}$

Domain $= \left\{x \in \mathbb{R} : x \neq \frac{3}{4}\right\}$ and range $= \{y \in \mathbb{R} : y \neq 0\}$

c)Horizontal asymptote, $y = 0$ and vertical asymptote, $x = \frac{1}{3}$

Domain $= \left\{x \in \mathbb{R} : x \neq \frac{1}{3}\right\}$ and range $= \{y \in \mathbb{R} : y \neq 0\}$

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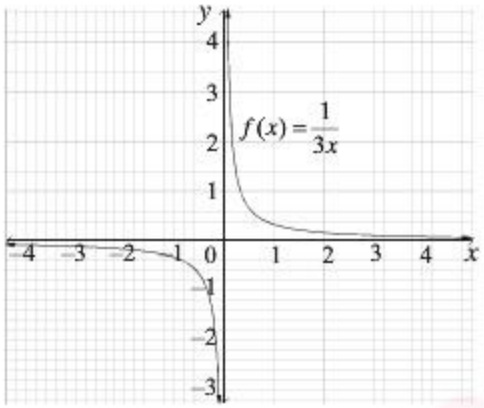
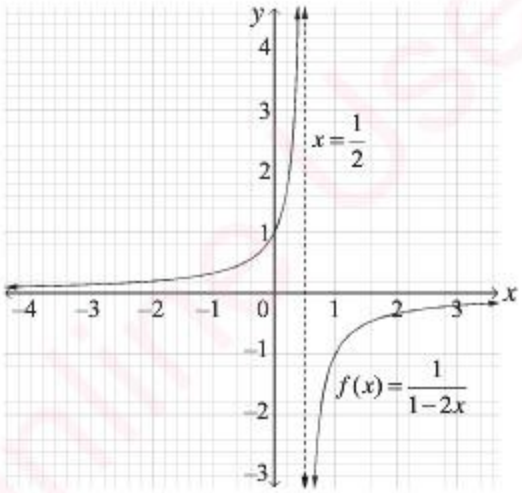
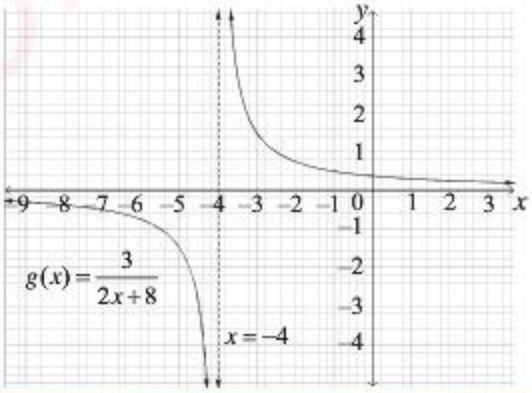
2. a)



b)



c)



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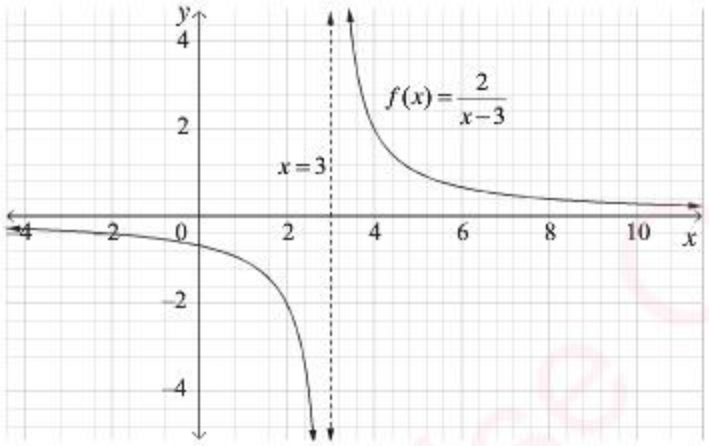
3. a) $f(x)$ is undefined at $x=3$



b) Domain $=\{x\in\mathbb{R}: x\neq 3\}$ and range $=\{y\in\mathbb{R}: y\neq 0\}$

4. a) $x=1$

b) $f(x)$ approaches positive and negative infinity.

c) $\left(0,\frac{3}{2}\right)$, no $x-$intercept

5. a) $a=-\frac{1}{8}$, $b=\frac{3}{8}$

b) Vertical asymptote, $x=3$ and horizontal asymptote, $y=0$

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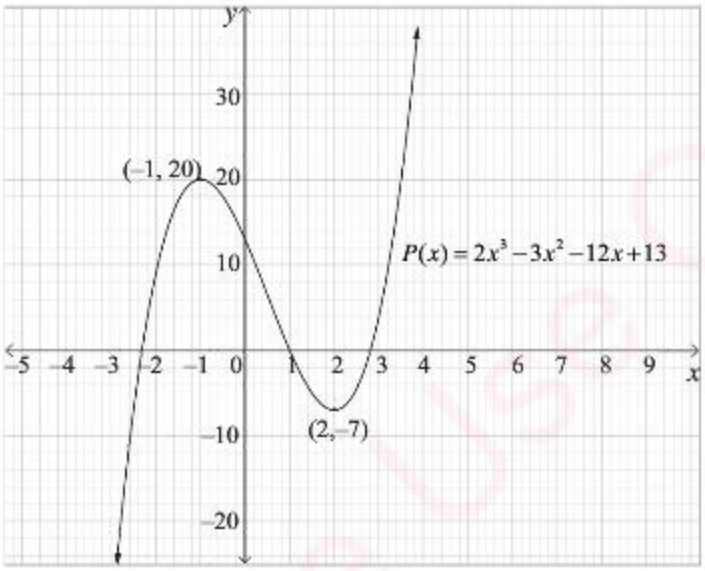
Revision Exercise 2

1. Graph



2. Factors: $(x-2)(x^2+2x+4)$

3. a) Degree 3
   b) Degree 3
   c) Degree 2
   d) Degree 4
   e) Degree 3

4. a) Not polynomial
   b) Polynomial
   c) Not polynomial
   d) Not polynomial

5. a) (i) Vertical assymptote $x=\frac{2}{3}$

   (ii) Horizontal assymptote $y=0$

   Horizontal assymptote: $y=0$

   b) Intercepts are

   $\left(0, -\frac{1}{2}\right)$ and no $x-$intercept

   c) Domain $=\left\{x \in \mathbb{R} : x \neq \frac{2}{3}\right\}$

   Range $=\{f(x) \in \mathbb{R} : y \neq 0\}$

6. a) $y=\{\pm 3, \pm 2\sqrt{2}, \pm\sqrt{6}, \pm 2\}$

   b) $y=\{3, 5, 21, 53\}$

   c) $y=\left\{1, \frac{1}{2}, \frac{1}{10}, \frac{1}{26}\right\}$

7. Range $=\{10, 17, 26, 37\}$

8. a) Domain $=\{x \in \mathbb{R} : x \leq 3\}$

   b) $f(-2)=\pm\sqrt{5}$

   $f(-1)=2$ or -2

   $f(1)=\pm\sqrt{2}$

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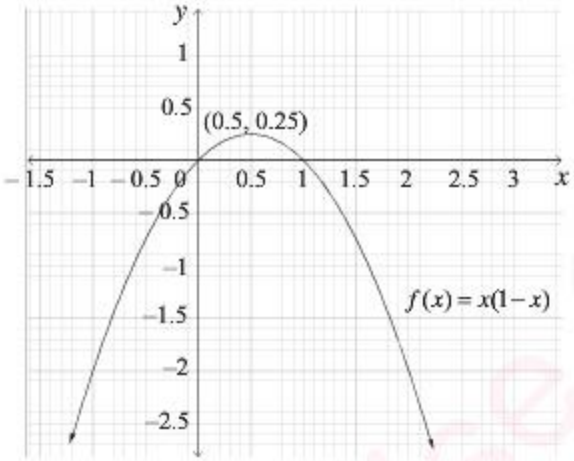
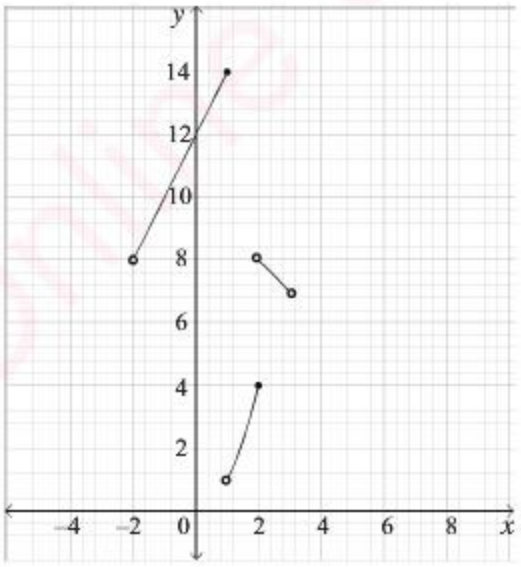
9. Graph :



Domain $=\{x : x \in \mathbb{R}\}$ and range $\left\{y \in \mathbb{R} : y \leq \frac{1}{4}\right\}$

10. a) Graph:



Domain $=\{x \in \mathbb{R} : -2 < x < 3\}$

Range $=\{y : 8 < y \leq 14, \quad 7 < y < 8, \quad 1 < y \leq 4\}$

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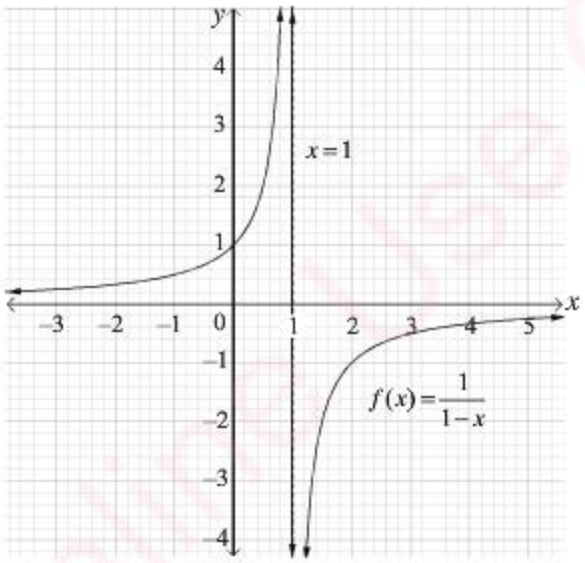
b) $f(1)=14$

$f(2)=4$

$f\left(2\frac{1}{2}\right)=7\frac{1}{2}$

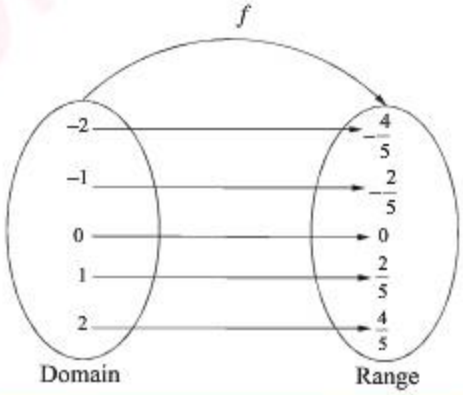
11. a) Because it makes a function to be undefined

b)



12. $a=7,\ b=1$

13.



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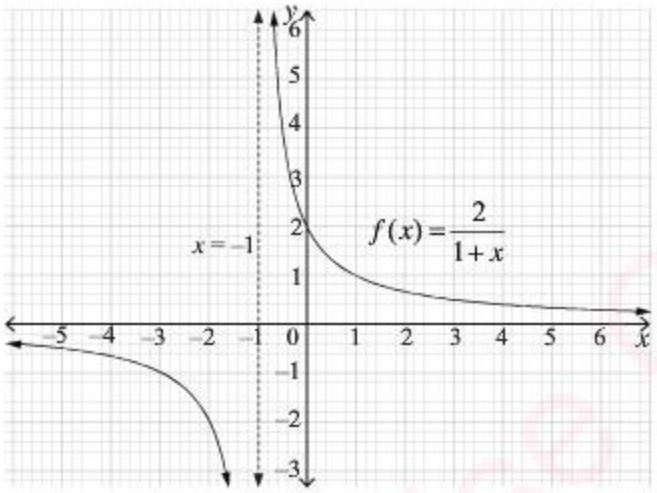
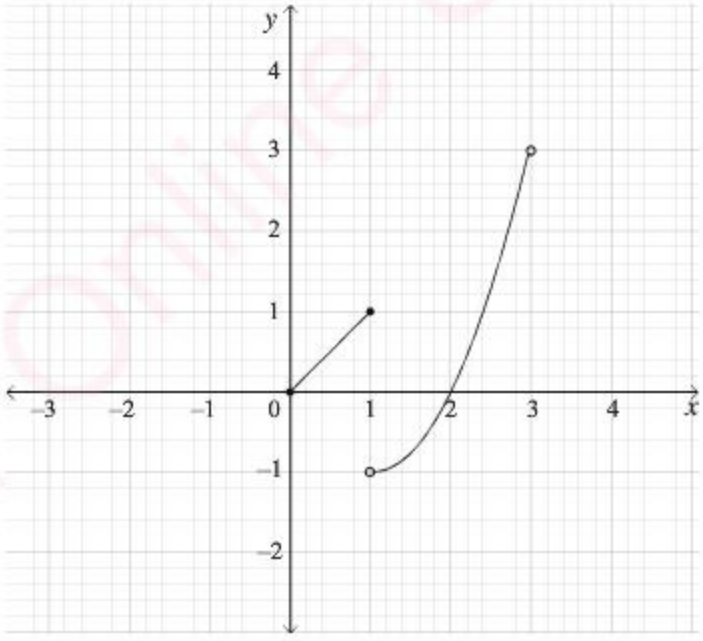


16. Graph of $f(x)$



Domain $= \{x \in \mathbb{R} : 0 \leq x < 3\}$ and range $= \{y \in \mathbb{R} : -1 < y < 3\}$

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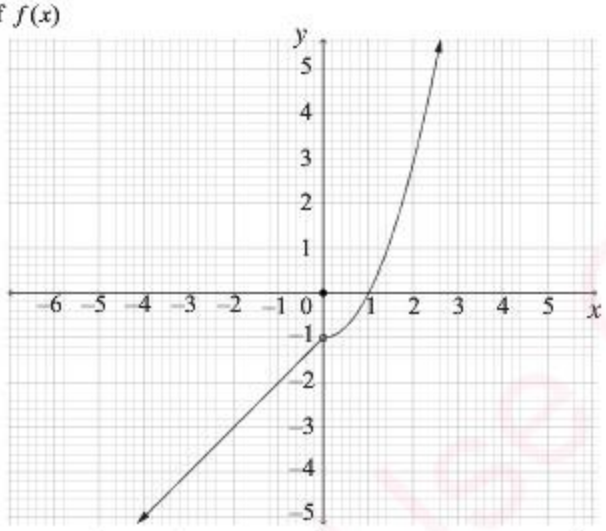
17. Graph of $f(x)$



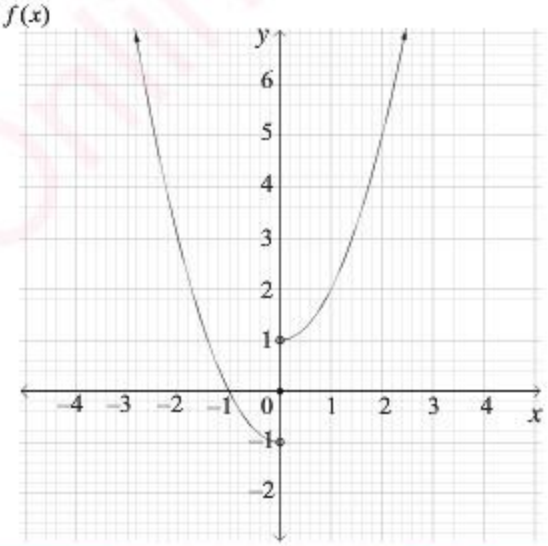
Hence: a) Domain $= \{x : x \in \mathbb{R}\}$ Range $= \{y \in \mathbb{R} : y = 0, y < -1, y > 1\}$

b) $f(-1) = -2$

$f(3) = 4$

$f(0) = 0$

18. Graph of $f(x)$



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a) Domain $= \{x \in \mathbb{R}\}$

Range $= \{y \in \mathbb{R} : y > -1\}$

b) $f(0) = 0$

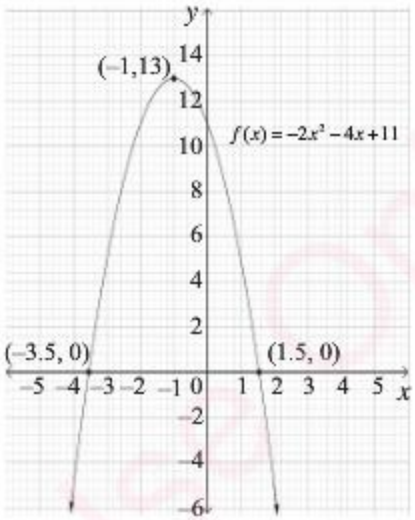
$f(-3) = 8$

$f(11) = 122$

19. a) Domain $= \{x \in \mathbb{R}\}$

Range $= \left\{y \in \mathbb{R} : y \geq \frac{11}{4}\right\}$

b) Turning point $= \left\{\frac{3}{2}, \frac{11}{4}\right\}$. Thus, it is a minimum turning point.



20. Factors: $x-1, \quad x-3, \quad x+2$

21. a) The turning point is (–1, 13).

Thus, it is a maximum turning point.

b) Intercepts:

x – Intercepts are (1.5, 0) and (–3.5, 0)

y – Intercept is (0, 11).

22. a) (i) $7x-1$

(ii) $8x^3 - 26x^2 - 67x - 15$

b) $3x - 10 + \frac{14}{x+1}$

c) $\frac{x+7}{x^2 - x - 2}$

CHAPTER THREE

Exercise 3.1

1. $(x, y) = (3, 2)$
2. $(x, y) = (17, -10)$
3. $(x, y) = (4, 2)$
4. $(x, y) = (5, -1)$
5. $(x, y) = \left(\frac{7}{3}, \frac{-2}{3}\right)$
6. $(x, y) = (10, 5)$
7. $(x, y) = (6.19,\ 0.81)$ or $(x, y) = (0.81,\ 6.19)$
8. $(x, y) = (7.5, 2.25)$ or $(x, y) = (6, 4.5)$
9. $x - y = 10, \quad x^2 - y^2 = 40, \quad (7, -3)$
10. $\begin{cases} 26x + 8y = 974,000 \\ 12x + 4y = 464,000 \end{cases}$

$(x, y) = (23{,}000,\ 47{,}000)$ Each box of books costs Tsh 23,000 and each box of coloured pencils costs Tsh 47,000 each .

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Exercise 3.2

1 a) Sequence, finite: countable number of terms
b) Sequence, finite: countable number of terms
c) Series, infinite: uncountable number of terms
d) Series, finite: countable number of terms

2. a) $1^2+2^2+3^2+4^2+5^2$ b) $2^3+3^3+4^3$ c) $2^2+3^2+4^2+\ldots+(1+n)^2$

3. a) $\sum_{r=1}^{n} 2r$ b) $\sum_{r=0}^{5} (-2)^r$ c) $\sum_{r=1}^{5} \frac{r}{3^{r-1}}$

5. ü

Exercise 3.3

1. a) $n=10$ b) $n=50$ c) $n=20$

2. a) 2601 b) 632 c) 420

3. a) $d=2$ b) $a_1=13$ c) $S_{10}=220$

4. $a_1=33,\ S_{16}=-72$ 5. 7650 6. 187

7. a) 15 b) $S_n=10$

Exercise 3.4

1. a) It is a GP, $S_{10}=1023$
b) It is an AP, $S_{10}=110$

2. $S_6=157.5$

3. a) $S_n=2\left(1-\frac{1}{2^n}\right),\ S_\infty=2$

b) $S_n=\frac{13}{90}\left(1-\frac{1}{10^n}\right),\ S_\infty=\frac{13}{90}$

c) $S_n=\frac{5}{9}\left(1-(0.1)^n\right),\ S_\infty=\frac{5}{9}$

d) $S_n=\frac{43}{99}\left(1-(0.01)^n\right),\ S_\infty=\frac{43}{99}$

e) $S_n=51.2\left(1-(-0.25)^n\right),\ S_\infty=51.2$

4. a) $\frac{2}{9}$ b) $\frac{7}{33}$ c) $\frac{47}{9}$ d) $\frac{445}{99}$ e) $\frac{1190}{333}$

5. a) $r=\frac{2}{3}$ b) $r_1=0.4$ or $r_2=0.6$
$G_1=60$ or $G_1=40$

7. a) $1+2+3+4$
b) $1+3+5+7+9$
c) $2+4+6+\ldots+200$
d) $y_1+y_2+y_3+\ldots+y_n$
e) $-1+2-3+4-\ldots$
f) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots$
g) $a+ar+ar^2+\ldots+ar^{n-1}$

8. a) $\sum_{n=1}^{\infty} 2n$
b) $\sum_{r=1}^{4} 2r-1$

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c) $\sum_{r=1}^{10} r^2$ e) $\sum_{r=1}^{n} \frac{r+1}{r}$ g) $\sum_{r=1}^{n} r(r+1)$

d) $\sum_{m=1}^{200} m$ f) $\sum_{j=1}^{5} k^j$ 9. 28.01 years

Exercise 3.5

1. Tsh 60,000, Tsh 90,000 and Tsh 150,000 2. $x=10$ 3. 4900 students
4. 2,000,000 Euros 5. 1:400 6. 100 litres
9. $A=1$ 10. $\frac{28}{3}$ 7. $x=-4$ or $x=4$ 8. $y=64$

Revision Exercise 3

1. a) $\begin{cases} x=2.234 \\ y=-0.076 \end{cases}$ or $\begin{cases} x=-1.8575 \\ y=-1.245 \end{cases}$

b) $\begin{cases} x=5 \\ y=3 \end{cases}$ or $\begin{cases} x=\bar{3} \\ y=\bar{5} \end{cases}$

2. a) $S_7=5465$

b) $S_{15}=-300$

c) $S_7=\frac{1093}{1458}$

d) $S_{12}=300$

3. a) $S_\infty=\frac{3}{2}$

b) $S_\infty=\frac{2}{3}$

c) $S_\infty=81$

4. $k=\frac{-1}{3}$

5. $P=2$ and $r=2$

6. $\frac{28}{65}$

7. 20 days

8. 25 Ω

9. 7.9 years

10. $x:y=-1:2$ or $x:y=-3:1$

11. a) $\begin{cases} y=3x+1 \\ y=7+2x-x^2 \end{cases}$

b) $\begin{cases} x=2 \\ y=7 \end{cases}$ *or* $\begin{cases} x=-3 \\ y=-8 \end{cases}$

12. a) $\sum_{i=1}^{n} \frac{1}{2^i}$

b) $\sum_{n=1}^{4} (2n-1)^2$

c) $\sum_{j=1}^{n} f_j x_j$

13. $y=25$

14. Shilla received Tsh 40,000,

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Mkumbo received Tsh 20, 000,

Eva received Tsh 80,000.

15. a) $x=10.5$ *b)* $x=3$ and $x=-\frac{4}{3}$

16. $y=3x$

17. $\frac{19}{2}$

CHAPTER FOUR

Exercise 4.1

1. 2
2. $6x-4$
3. $2ax+b$
4. $-9x^2$
5. $3x^2+12x+4$
6. $3x^2+12x+9$
7. $\frac{6}{5}x+\frac{7}{5}$
8. $-\frac{4}{5}x+\frac{1}{5}$
9. $\frac{-1}{(x+1)^2}$
10. $\frac{1}{\sqrt{2x+1}}$

Exercise 4.2

1. $-8x+6$
2. $3x^2-4x-5$
3. $-4x-5$
4. $\frac{-6}{x^7}+6x$
5. $\frac{2\sqrt{x}}{x}-4$
6. $-x^{-2}-8x^{-3}-9x^{-4}$
7. $\frac{1}{\sqrt{x}}-\frac{2}{\sqrt{x^3}}$
8. $-\frac{1}{x^2}+5$
9. $\frac{-1}{2\sqrt{x^3}}$
10. $-\frac{2}{x^3}-\frac{4}{x^5}-\frac{6}{x^7}$

Exercise 4.3

1. $6(3+2x)^2$
2. $-\frac{4x}{(2x^2+4)^2}$
3. $\frac{-1}{\sqrt{(2x-3)^3}}$
4. $\frac{-3v^2}{2\sqrt{(1+v^3)^3}}$
5. $\frac{8-6z^2}{(z^3-4z)^3}$
6. $72x^2(2-x^3)^{-7}$

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7. $\dfrac{-15x}{(3+x^2)^{\frac{5}{2}}}$ 8. $\dfrac{1}{3\sqrt[3]{(1+x)^4}}$ 9. $\dfrac{1-4x\sqrt{x}}{\sqrt{x}\left(x^2-\sqrt{x}\right)^8}$

10. $\dfrac{5}{3}\left(9x^2+\dfrac{1}{x^2}\right)\left(3x^3-\dfrac{1}{x}\right)^{\frac{2}{3}}$

Exercise 4.4

1. $16x-6$ 2. $4x^3-6x^2+2x$ 3. $-4x^3-18x^2-16x+6$

4. $\left(5x^2+34x+56\right)(1+x)^2$ 5. $\dfrac{-3x^2-8x+1}{2\sqrt{(x+4)(1-x^2)}}$ 6. $\dfrac{-12x}{\sqrt{1-4x}}$

7. $4x^3+12x^2+6x-4$ 8. $(72x^2-20x-8)(3x+5)^3$ 9. $6x-\dfrac{3}{2}\sqrt{\dfrac{x}{2}}$

10. $\dfrac{-12x^3+2x}{\sqrt{3-4x^2}}$

Exercise 4.5

1. $\dfrac{-12}{(1+2x)^3}$ 2. $\dfrac{1}{(x+1)^2}$ 3. $\dfrac{-x^2+6x-4}{(3-x)^2}$

4. $\dfrac{x-2}{2(1+2x)^3}$ 5. $\dfrac{-1}{\sqrt{(1+2x)^3}}$ 6. $\dfrac{20x+60}{(1-3x)^3}$

7. $\dfrac{-4}{\sqrt{(x+4)(x-4)^3}}$ 8. $\dfrac{4}{\sqrt{2x}\left(2-\sqrt{2x}\right)^2}$ 9. $\dfrac{x+8}{2(x+4)^{\frac{3}{2}}}$

10. $\dfrac{6x+2x^3}{\left(3+2x^2\right)^2}$

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Exercise 4.6

1. $18x$
2. $6x-6$
3. $a(n^2-n)x^{n-2}$
4. $6-24x$
5. $48x+96$
5. $20x^3-12x$
7. $40x^3+48x^2+12x$
8. $\dfrac{9x-12}{4\sqrt[4]{(1-x)^3}}$
9. $-\dfrac{2}{9}(x+2)^{-\frac{5}{3}}$
10. $\dfrac{-2}{(1+x)^3}$

Exercise 4.7

1. $\dfrac{2x+2y}{1-2x-2y}$
2. $\dfrac{26x-5x^4y-y^2}{2xy+x^5+12y^2}$
3. $\dfrac{x-9y}{9x+4y^3}$
4. $\dfrac{20x-y^5}{5xy^4+8}$
5. $\dfrac{y-6x}{2y-x}$
6. $\dfrac{3x^2-y^4}{4xy^3-2}$
7. $-\dfrac{2xy^3+8y}{3x^2y^2+8x}$
8. -1
9. $\dfrac{2x^{-3}-2x}{3y^2-1}$
10. $\dfrac{x+y+1}{2y-x}$

Exercise 4.8

1. 2
2. –1
3. 0
4. $\dfrac{1}{4}$
5. $\dfrac{11}{81}$
6. (1, –8) or (–1, –12)
7. 3
8. 0
9. 7
10. $-\dfrac{29}{4}$

Exercise 4.9

1. $\dfrac{dx}{dy}=\dfrac{1}{5}$, $\dfrac{dy}{dx}=3$
2. 10 cm^2 / s
3. 0.03 cm^3 /s
4. 1.14 mm / s
5. 0.057 cm / s

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Exercise 4.10

1. 15 m /s; 12 m / s^2
2. 29.4 m/s
3. $v(t) = -4t + 10$; $a(t) = -4$
4. 5 seconds
5. −86m/s

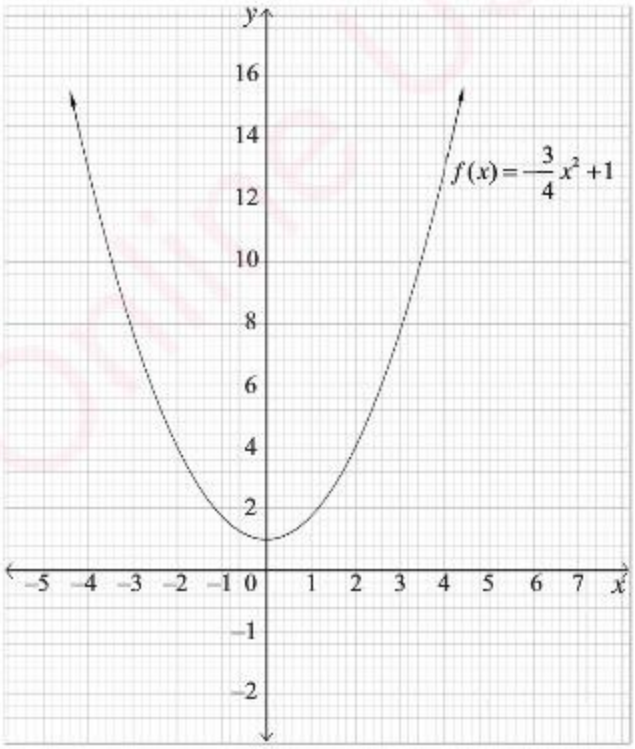
Exercise 4.11

1.

a) Maximum value of $f(x) = \frac{26}{27}$, Minimum value of $f(x) = -18$. The point $\left(\frac{-2}{3}, \frac{26}{27}\right)$ is a maximum turning point and point (2, −18) is a minimum turning point.

b) Minimum value of $f(x) = -\frac{1}{4}$ point $\left(-2, -\frac{1}{4}\right)$ is a minimum point. Maximum value of $f(x) = \frac{1}{4}$, point $\left(2, \frac{1}{4}\right)$ is a maximum point.

3. a) Graph:



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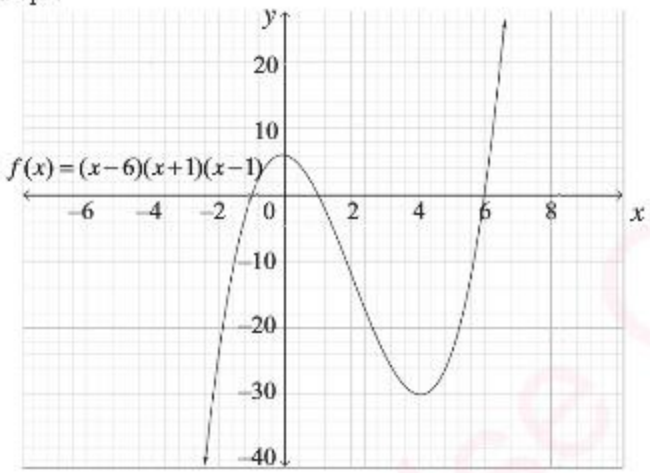
b) Graph:



4. Dimensions are:
12.6 cm by 12.6 cm by 12.6 cm

5. radius = 1.996 m, height = 7.996 m

Exercise 4.12

1. a) Tsh 1,200,008,000 b) Tsh 1,200,004
2. a) $P(x) = -4x^2 + 18x - 20$ b) $R'(x) = 22 - 8x$
3. a) $R(x) = \frac{100000x}{50x - 100} - 50x$ b) Tsh 2,000
4. a) Tsh 64 b) Tsh 44 c) 0
5. a) Tsh 264 b) Cost will increase by Tsh 306

Revision Exercise 4

1. a) $12x^2 + 7$ b) $28x^6$ c) $-\frac{12}{x^4}$
2. a) $\frac{4x}{x^4 + 2x^2 + 1}$ b) $\frac{-5}{x^2 - 6x + 9}$ c) $\frac{-3}{x^2 + 4x + 4}$
3. a) $32x(2x^2 + 7)^7$ b) $15x^2 + 4x + 15$ c) $\frac{4x + 9}{2\sqrt{2x^2 + 9x + 3}}$
   d) $(3x + 5)^3(72x^2 - 20x - 8)$

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4. a) $8x^3+6x$ b) $24x^3-36x^2+6x+3$ c) $\dfrac{-9x^{-4}+24x^{-3}-5x^{-2}}{\left(x^2-6x+3\right)^2}$

d) $\dfrac{-6x^2+26x-18}{\left(x^2+6x-16\right)^2}$

5. a) 1 b) -24

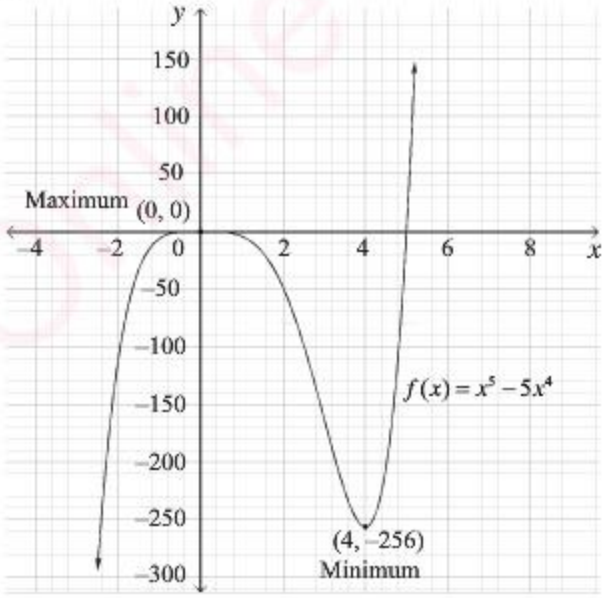
6. a) $y=0$; gradient $=3$ b) $y=1$; gradient $=-\dfrac{1}{2}$

$y=8$; gradient $=14$ $y=-\dfrac{1}{2}$; gradient $=-\dfrac{1}{8}$

7. a) 3 b) $2x+2$ c) $-15x^2-10x-8$

d) $(x-1)^2\left(5x^2-2x+6\right)$ f) $12x^3+\dfrac{1}{\sqrt{x}}$ e) $12x-16$

8. a) Graph:



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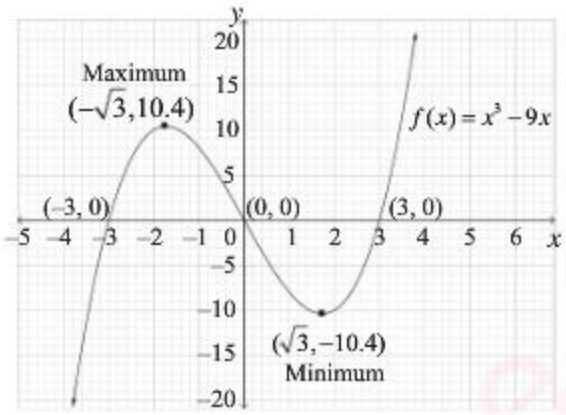
b) Graph:



9. Velocity $=3t^2+4t-4$

   Acceleration $=6t+4$

10. $\frac{dh}{dt}=-0.15$cm/s (i.e. it is falling at 0.15 cm / s)

11. Speed 0.99 m/s away from the wall

12.

a) Marginal cost function $C'(x)=36x-10$

b) Total cost $=92$ units when $x=2$

c) The maximum cost is 38.61 units

CHAPTER FIVE

Exercise 5.1

1. $\frac{x^4}{4}+c$
2. $\frac{5x^2}{2}+c$
3. $\frac{-1}{3x}+c$
4. $\frac{a^2x^8}{8}+c$
5. $\frac{2x^{\left(\frac{3}{2}\right)}}{3}+c$
6. $2\sqrt{u^5}+c$
7. $3\sqrt[3]{x^2}+c$
8. $\frac{40}{11}x^{\left(\frac{11}{8}\right)}+c$
9. $x+c$
10. $2t\sqrt{y}+c$

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Exercise 5.2

1. $\frac{5}{2}x^2-7x+c$
2. $4x^2+\frac{5}{2x^2}+c$
3. $\frac{7}{32}(4x)^8+c$
4. $-\frac{1}{6x^3}+c$
5. $4t^{\frac{3}{2}}-\frac{14}{3}t^{\frac{3}{2}}-24t^{\frac{1}{2}}+c$

Exercise 5.3

1. $\frac{3}{28}\left(x^4-7\right)^7+c$
2. $\frac{2}{9}\left(\sqrt{x^3+16}\right)^3+c$
3. $\frac{1}{12}\left(x^2-6\right)^6+c$
4. $\frac{1}{56}\left(x^7-4\right)^8+c$
5. $\frac{1}{5}\left(x^3+3x-10\right)^5+c$
6. $\frac{1}{2}\sqrt{x^2+3x-2}+c$

Exercise 5.4

1. 6
2. 0
3. 35
4. 822.07
5. 110.833
6. 18
7. 13.599
8. 1.32×10^{12}
9. 3.51
10. $\frac{2}{3}(\sqrt{7}-1)$
11. $22\frac{2}{3}$

Exercise 5.5

1. $8\frac{2}{3}$ Square units
2. Graph

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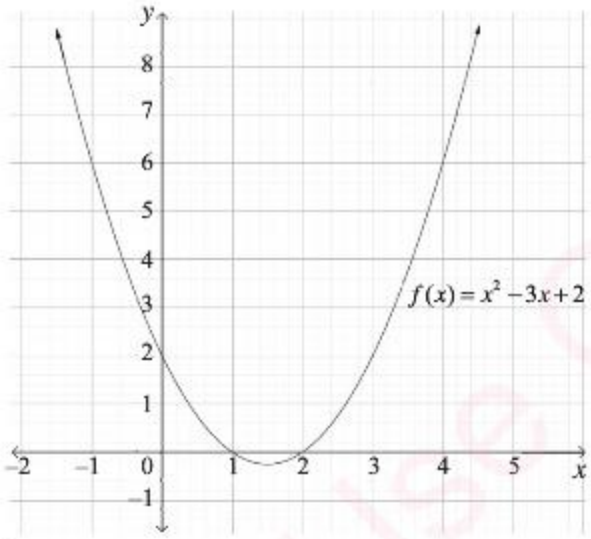

Area $= 1\frac{5}{6}$ Square units

3. 3.60 Square units
4. 4.5 Square units
5. $5\frac{4}{5}$ Square units

Exercise 5.6

1. 13.5 Square units
2. 10.67 Square units
3. $\frac{1}{3}$ Square units
4. 14.9 Square units
5. $\frac{9}{4}$ Square units

Exercise 5.7

1. 20π Cubic units
2. $\frac{8}{3}\pi$ Cubic units
3. $25\frac{3}{5}\pi$ Cubic units
4. 55π Cubic units
5. 48π Cubic units
6. $\frac{2}{3}\pi$ Cubic units
7. $\frac{4}{3}\pi r^3$ Cubic units

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Exercise 5.8

1. (*b*) $C(x)=\frac{450}{7}x+c$
2. Tsh 128,529.20
3. $C(x)=3000x-25x^2+\frac{5}{3}x^3+44087$, Average cost $=3000-25x+\frac{5}{3}x^2+\frac{44087}{x}$
4. $R(x)=30x+x^2+16$
5. 333.33

Exercise 5.9

1. -4.5 metres
2. 2 units
3. -14.7 metres
4. 20 km
5. 74 units
6. 206.13 m/s
7. $s(t)=t^2+6t^3$

Revision exercise 5

1. a) $\frac{x^6}{6}+c$ b) $\frac{4}{7}x^{\left(\frac{7}{4}\right)}+c$ c) $-\frac{1}{x}+c$

   d) $2x^{\left(\frac{3}{2}\right)}+\frac{1}{x^2}-\frac{1}{x}+c$ e) $\frac{2}{3}x^{\left(\frac{3}{2}\right)}-\frac{9}{5}x^{\left(\frac{5}{3}\right)}+6x+c$

2. $y=\frac{x^2}{2}+x-1$
3. $C(x)=3x+20x^2-\frac{5}{3}x^3+45$
4. $f(x)=\frac{x^3}{3}+x-10$
5. a) $A=0.1667$ square units
   b) $A=21.33$ Square units
6. a) $\frac{x^2}{2}-4x+c$ b) $\frac{2}{9}\left(3x^2+7\right)^{\frac{3}{2}}+c$ c) $\frac{1}{18}(2x+1)^9+c$

   d) $\frac{7}{4}x^4-\frac{4}{3}x^3+x^2-x+c$ e) $\frac{1}{6}x^3+2x^2-2x+c$ f) $\frac{(x^3-4)^7}{21}+c$
7. $C(x)=0.5x^3-0.7x^2-45x+c$

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8. a) $\frac{(t^3+2t-10)^4}{4}+c$ b) $\frac{1}{24}(4t-1)^6+c$ c) $\frac{1}{32}(t^4-3)^8+c$

9. a) 28,153,218.13 b) 2.472

10. a) 30.667 b) 4.667

11. a) $A=0.167$ square units b) $A=21.33$ square units

12. $n=\frac{2}{3}$ 13. Volume 27π cube units 14. $R(x)=5x+6x^2+\frac{16}{3}x^3+c$

15. $\frac{4}{15}$ units 16. 82.17 metres

CHAPTER SIX

Exercise 6.1

1. Frequency distribution table

Class intervals	Tally	Frequency
1	////	4
2	𝍸 ////	9
3	𝍸 //	7
4	𝍸 //	7
5	///	3
6	//	2

2. Frequency distribution table

Class intervals	Tally	Frequency
1 – 5	𝍸 ////	9
6 – 10	𝍸 ////	9
11 – 15	𝍸 /	6
16 – 20	////	4
21 – 25	///	3

3. Frequency distribution table

Value	Tally	Frequency
40	/	1
43	/	1
47	//	2
48	/	1
50	//	2
51	//	2
52	/	1
75	/	1

4. Frequency distribution table

Value	Tally	Frequency
1	𝍸 /	6
2	/	1
4	/	1
5	////	4
6	///	3
7	////	4
8	//	2
9	//	2

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5. Frequency distribution table

Class intervals	Class mark	Frequency
91 – 100	95.5	9
81 – 90	85.5	6
71 – 80	75.5	3
61 – 70	65.5	0
51 – 60	55.5	2

Exercise 6.2

1. Frequency distribution table

Class intervals	Tally	Frequency
1 – 9	////	4
10 – 18	𝍸 //	7
19 – 27	𝍸 𝍸 𝍸 /	16
28 – 36	𝍸 𝍸 ///	13
37 – 45	𝍸 //	7
46 – 54	//	2
55 – 63	/	1

2. Frequency distribution table

Score	Tally	Frequency	Cumulative frequency
2	///	3	3
3	////	4	7
4	////	4	11
5	//	2	13
6	///	3	16
7	//	2	18
8	//	2	20
9	𝍸 ///	8	28
10	////	4	32
11	𝍸	5	37
12	///	3	40

3. a) 25 b) 10 c) 100 – 109

4. Cumulative distribution table

Class intervals	Tally	Frequency	Cumulative frequency
20 – 21	𝍸	5	5
22 – 23	/	1	6
24 – 25	𝍸 //	7	13
26 – 27	///	3	16
28 – 29	//	2	18

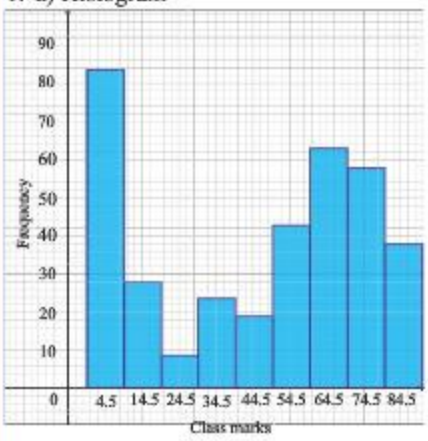
5. a) i) $y = 7$ ii) 4

b)

Class intervals	Frequency	Cummulative frequency
33 – 36	5	5
37 – 40	7	12
41 – 44	11	23
45 – 48	12	35
49 – 52	15	50

Exercise 6.3

1. a) Histogram



b) Children and old people are susceptible to deseases.

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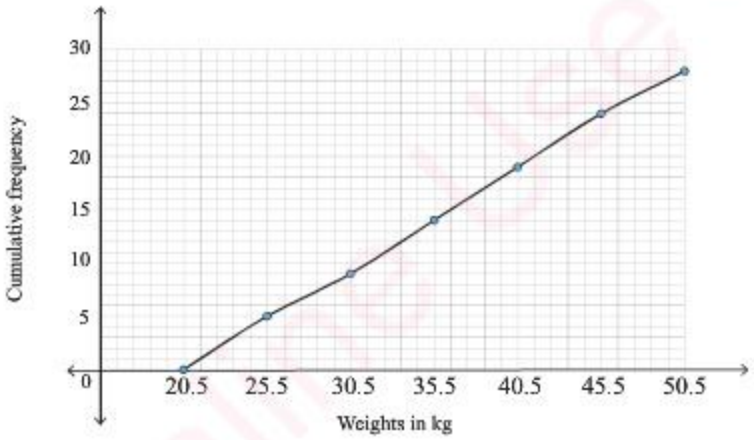
2. a) The cummulative frequency distribution

Class interval	Tally	Frequency	Cumulative frequency
21 – 25	𝍸	5	5
26 – 30	////	4	9
31 – 35	𝍸	5	14
36 – 40	𝍸	5	19
41 – 45	𝍸	5	24
46 – 50	////	4	28

b) A cumulative frequency curve (Ogive)



3. a) Frequency distribution table

Class Interval	Tally	Frequency
30 – 49	///	3
50 – 69	𝍸 /	6
70 – 89	///	3
90 – 109	//	2
110 – 129	/	1
130 – 149	//	2

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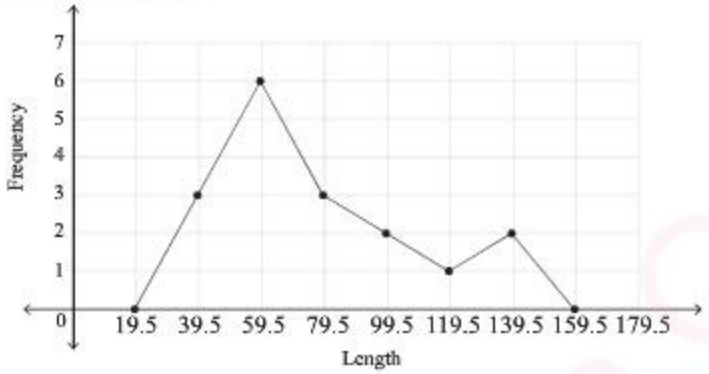
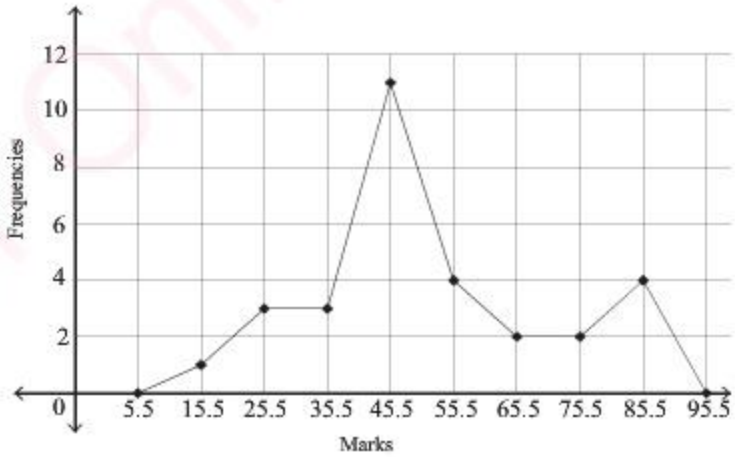
b) A frequency polygon:



4. a) Frequency distribution table

Class interval	Tally	Frequency
11 – 20	/	1
21 – 30	///	3
31 – 40	///	3
41 – 50	𝍸 𝍸 /	11
51 – 60	////	4
61 – 70	//	2
71 – 80	//	2
81 – 90	////	4

a) Frequency polygon



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5. Pictorial representation of planted trees

Key: represents 10 trees

2015	
2016	
2017	
2018	
2019	

Exercise 6.4

1. a) 9 b) 29.9
2. a) 5 b) 55
3. a) 50.78 b) 50.78
4. 15.3 5. 32

Exercise 6.5

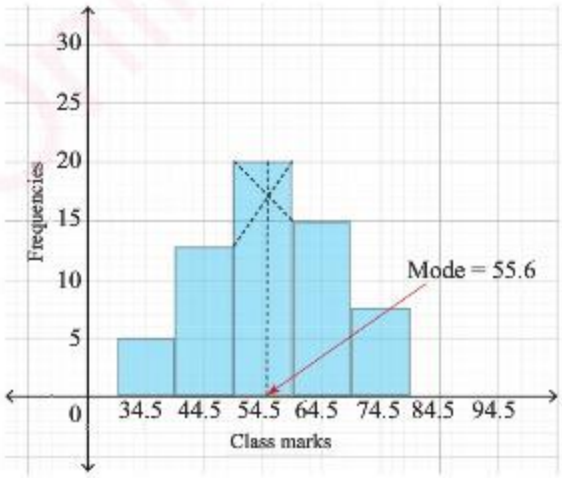
1. 50 2. 4 days 3. 36.8 years
4. a) 10 b) 49 5. 47.15

Exercise 6.6

1. a) Median 7, Mode 2
   b) median is 20, modes are 10, 20
2. a) 12.98 b) 10 c) 12
3. a) 20 b) 55.75 c) 54.74
4. a) 10 b) 49
5. a) 10-14 b) 12.5

Exercise 6.7

1. a) 26 b) 42, 17 c) 25 d) 12.5
2. 23, 20, 25.5, 5.5
3. a) 168.88 b) 164.69 c) 2.69
4. a) 10.54, 33.22, 12.24, 27.29
   b) 48.08, 63.09, 49.33, 59.35
5. a) mode ≈ 55.7
   b) $Q_1 \approx 47.83$,
   Median$(Q_2) \approx 56$ and
   $Q_3 \approx 64.8$



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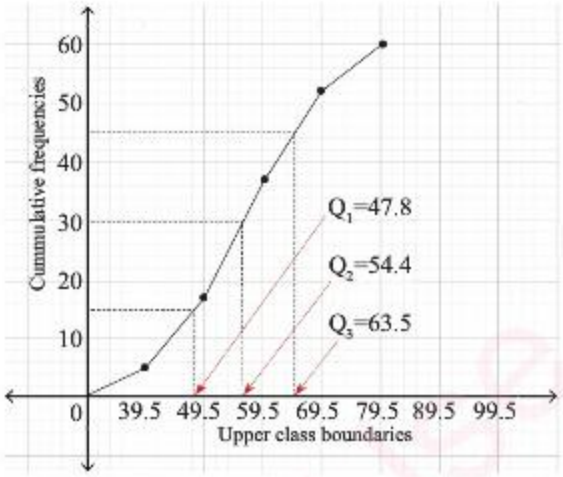

Exercise 6.8

1. a) 48 b) 80
2. a) 6.67 , 2.58 b) 30, 5.48
3. a) 2.16, 1.47 b) 8.01, 2.83
4. a) 10, 10, 10 b) 2, 0, 8.05
   c) Data set C has the largest standard deviation
   d) Yes, since data set C has data value that are far away from the mean compared to set A and set B
5. 43.5, 5.55
6. 2.24
7. A: 1.06, 1.313
   B: 1.2, 1.31
4. a) 13.71, 23, 34.39
   b) 20.68
5. 16.125, 10.88, 6.38, 25.21
6. a) $\approx 19.8, \approx 34.5$
   b) ≈ 34.3
7. 29.52, 30.17, 31.11
8. 2.8, 1.1225
9. a) 1.0782, 1.1624
   b) 201.81, 201.61
10. Frequency distribution table

Class intervals	Tally	Frequency
1 – 10		0
11 – 20	𝍸 ////	9
21 – 30	///	3
31 – 40	𝍸 /	6
41 – 50	///	3

Revision Exercise 6

1. 28.52
2. a) 22.04, 8.15
   b) 19.76, 16.76
3. 16.64, 14.625, 16.85, 18.82

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a) 26.93 b) 16.33, 25.5, 36.75 c) 10.21 d) 18.67

11. 7.94
12. 2.1807

CHAPTER SEVEN

Exercise 7.1

1. a) 24 words b) 120 words c) 5,040 words
2. a) 256 b) 24
3. a) 1900 b) 7.0690
4. a) 1440 ways b) 3600 ways

Exercise 7.2

1. a) 10 b) 24 c) 19,958,400
   d) 7.7556×10^{24}
2. a) 3,326,400 ways b) 360 ways c) 420 ways
   d) 15,120 ways
3. 60 ways
4. 24 ways
5. 15 routes
6. 520 ways
7. 60,480 ways
8. 120 numbers
9. 180 words
10. 45360 ways.

Exercise 7.3

1. 504 ways
2. 30 ways
3. 336 ways
4. 35
5. 220
6. 55

Exercise 7.4

1. $S=\{TA, TC, TK, AT, AC, AK, CT, CA, CK, KT, KA, KC\}$
2. $S=\{(boy,girl), (girl,boy), (girl,girl), (boy,boy)\}$
3. $P(\text{even})=\frac{1}{2}$
4. a) $P(\text{odd})=\frac{5}{11}$ b) $P(\text{even})=\frac{6}{11}$
5. $P(\text{blue})=\frac{3}{5}$

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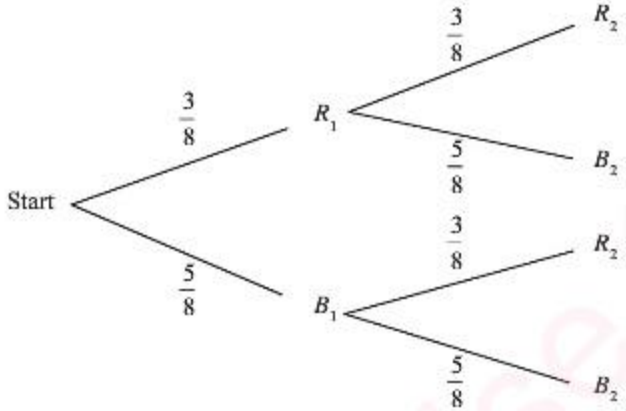
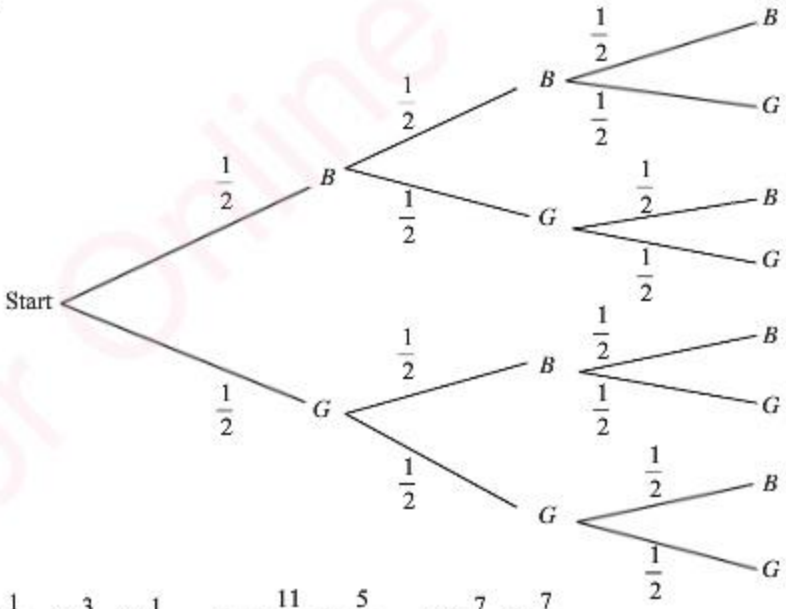
Exercise 7.5

1. a)



b) $\frac{15}{64}$ c) $\frac{15}{32}$ 2. a) $\frac{1}{6}$ b) $\frac{1}{4}$

3.



a) $\frac{1}{8}$ b) $\frac{3}{8}$ c) $\frac{1}{2}$ 4. a) $\frac{11}{36}$ b) $\frac{5}{36}$ 5.a) $\frac{7}{15}$ b) $\frac{7}{15}$

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Exercise 7.6

1. (a) $\frac{8}{13}$ (b) $\frac{10}{13}$

2. $P(E_1 \text{ or } E_2) = \frac{2}{9}$, $P(E_1 \text{ or } E_3) = \frac{1}{6}$

3. $\frac{7}{26}$

4. Sample space = 7, since the total number of marbles is 7.

5. a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) $\frac{1}{6}$

6. a) $\frac{1}{2}$ b) $\frac{1}{8}$ c) $\frac{3}{8}$ d) $\frac{5}{8}$

Exercise 7.7

1. a) $\frac{4}{25}$ b) $\frac{12}{25}$ 2. $\frac{1}{4}$

Exercise 7.8

1. a) $\frac{15}{28}$ b) $\frac{15}{32}$ 2. $\frac{1}{17}$ 3. $\frac{1}{6}$

4. a) $\frac{1}{8}$ b) $\frac{5}{8}$ 5. 0.9048

Exercise 7.9

1. a) 0.54 b) 0.22 2. a) 0.8485 b) 0.4455

3. 0.6 4. 0.7805 5. 0.7711

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Revision Exercise 7

1. 96 2. 17,280 3. 60
4. a) 120 b) 646,646 c) 1 d) 39,916,800 e) 32,760
5. a) 7 b) 3 6. 90% 0r 0.9 7. $\frac{1}{8}$ 8. A and B are mutually exclusive
9. $\frac{17}{35}$ 10. 0.1 11. 0.1279 12. a) 0.455 b) 0.895

CHAPTER EIGHT

Exercise 8.1

1. a) 248° is negative; lies in the 3rd quadrant b) 320° is positive; lies in the 4th quadrant
   c) 140° is negative; lies in the 2nd quadrant
2. a) $\sin 18°$ b) $\tan 71°$ c) $\cos 14°$
3. a) $-\text{cosec}35°$ b) $-\cot 50°$ c) $\sec 25°$
4. a) Third quadrant b) Fourth quadrant c) Fourth quadrant

Exercise 8.2

1. a) $\sqrt{3}$ b) $-\frac{\sqrt{2}}{2}$ c) $-\frac{\sqrt{3}}{2}$ 3. a) $\sqrt{3}$ b) $-\frac{2\sqrt{3}}{3}$ c) $\sqrt{2}$ 4. $\frac{\sqrt{3}}{2}$
5. a) 7 b) $\frac{7+4\sqrt{3}}{3}$

Exercise 8.3

1. $R\hat{P}Q = 25.89°$ and $R\hat{P}Q = 106.11°$
2. $\overline{DF} = 7.18cm$
3. $C\hat{A}B = 58.99°$ and $A\hat{C}B = 81.01°$
4. 11.11 cm
5. 11.18 cm
6. No; because an angle opposite to the side $\overline{AB}$ or $\overline{AC}$ is not known.

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Exercise 8.4

1. 26.38° 2. 8.78 cm 3. 27.28° 4. 25.21° 5. 40.45°, 52.99° and 86.55°

6. 11.128 kilometres

Exercise 8.5

1. a) $-\frac{3}{5}$ b) $-\frac{44}{117}$

2. a) $\frac{336}{625}$ b) $\frac{527}{625}$ c) $\frac{336}{527}$

3. a) $\frac{\sqrt{6}+\sqrt{2}}{4}$ b) $\frac{\sqrt{3}}{2}$ c) $2+\sqrt{3}$

4. a) $-k$ b) $-\frac{1}{\sqrt{1-k^2}}$ or $-\frac{\sqrt{1-k^2}}{1-k^2}$ c) $\frac{k+\sqrt{1-k^2}}{k-\sqrt{1-k^2}}$

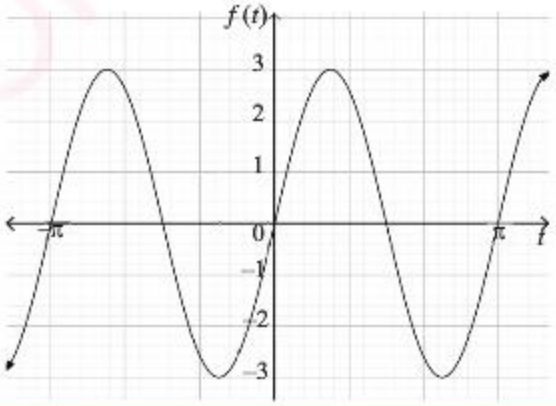
5. a) $y=\frac{2x}{1-x^2}$ b) $y^2=-2x$ c) $x=2y-1$

7. a) 56.4°, 123.6°, 270° b) 270°, 330° c) $-\frac{7\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$

d) $-\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

Exercise 8.6

3. The amplitude is 3, the period π



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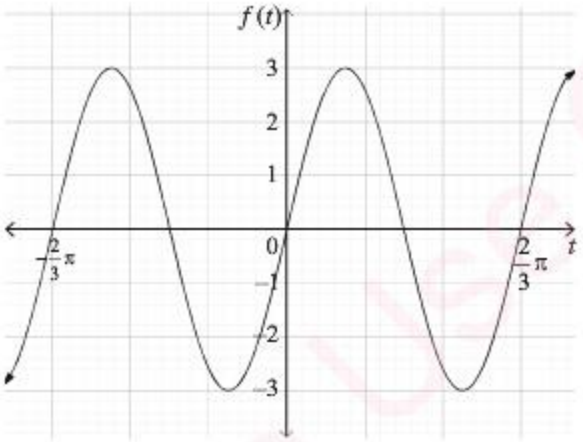
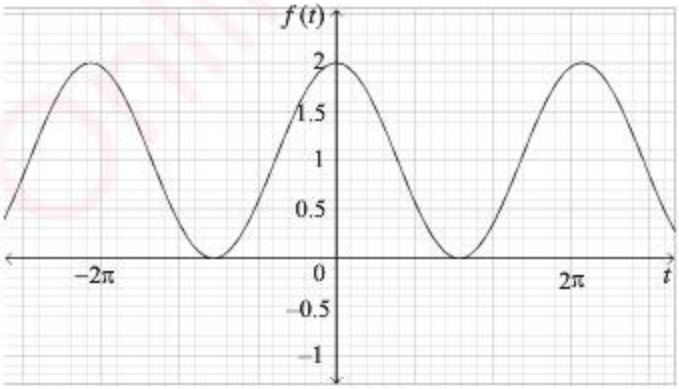
4. a) Odd function, because $f(-x) = -f(x)$ b) Odd function, because $f(-x) = -f(x)$

c) Neither even nor odd, as $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

5. a) The amplitude is 2, the period $\frac{2\pi}{3}$



b) The amplitude is 1, the period 2π



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Exercise 8.7

1. $3\cos 3x$ 2. $-2\sin 2\theta$ 5. $\sec\theta\tan\theta$

6. a) $2\sec^2 2x$ b) $5\sec(1+5x)\tan(1+5x)$

7. a) $(2x+4)\cos(x^2+4x+5)$ b) $\dfrac{\cos x\sqrt{1+\sin x}}{2(1+\sin x)}$

9. a) $-20\sin 4x\cos^4 4x$ b) $-6\cot 3\theta\ cosec^2 3\theta$

10. a) $-\dfrac{1}{1+\sin\theta}$ b) $3\cos 3\theta\sec 2\theta+2\tan 2\theta\sin 3\theta\sec 2\theta$

Exercise 8.8

1. a) $-\dfrac{1}{7}\cos(7x+4)+c$ b) $-\dfrac{1}{3}\sin^3\theta+\sin\theta+c$ c) $-\dfrac{1}{4}\cot\theta+c$

d) $\dfrac{5}{2}\sin(2x-7)+c$ 2. a) 0.2854 b) $\dfrac{1}{6}$ c) 1 d) 0.0678

4. 0.1609

Revision Exercise 8

1. a) $x=\dfrac{2y}{1-y^2}$ b) $y^2=16-8x$

3. a) $\cot 2\theta$ b) 0 c) $\tan 6\theta$

4. a) $-155.7°, -114.3°, 24.3°, 65.7°$ b) $0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, 2\pi$ c) $-\pi, -\dfrac{2\pi}{3}, 0, \dfrac{2\pi}{3}, \pi$

5. a) $31.69°$ b) $82.71°$

7. a) $-3\cos^2\theta\sin\theta$ b) $\dfrac{3}{2}\sec^2\dfrac{3}{2}x$ c) $\dfrac{2\cos 2x\cos 5x+5\sin 2x\sin 5x}{\cos^2 5x}$

d) $2\sin x\cos x\tan 3x+3\sin^2 x\sec^2 3x$

8. a) $-3\sin 3x$ b) $\dfrac{1}{3}\cos\dfrac{1}{3}\theta$

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9. a) $-\cos\left(\frac{1}{2}x+3\right)+c$ b) $-2\cos\sqrt{x}+c$ c) $-\frac{1}{2}\operatorname{cosec}^2\theta+c$ d) $-\frac{1}{6}\operatorname{cosec}^3\theta+c$

10. a) -0.098 b) 1.5774 c) 0.143 d) 0.063

CHAPTER NINE

Exercise 9.1

1. a) Base 3, exponent x
   b) Base b, exponent $x+2$
   c) Base e, exponent $2x+3$

2. a) p^7
   b) a^{10}
   c) x^{3k+5}

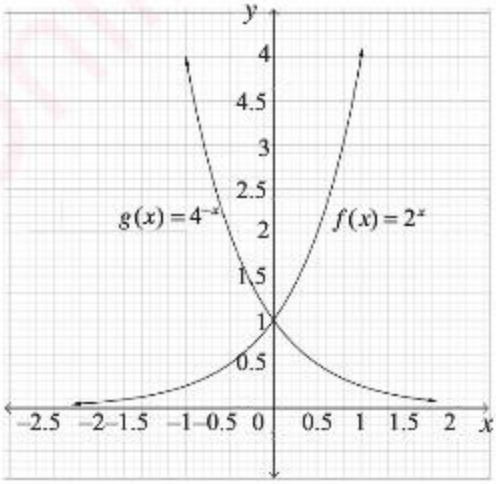
3. a) $-67{,}108{,}864$
   b) 16336
   c) -4

4. a) 4
   b) p^{2q-3}
   c) x^{2p+4q}

5. a) $\frac{-4}{7}$
   b) 1
   c) 1, -2

Exercise 9.2

1. b) The graph of $f(x)=2^x$ and $g(x)=4^{-x}$



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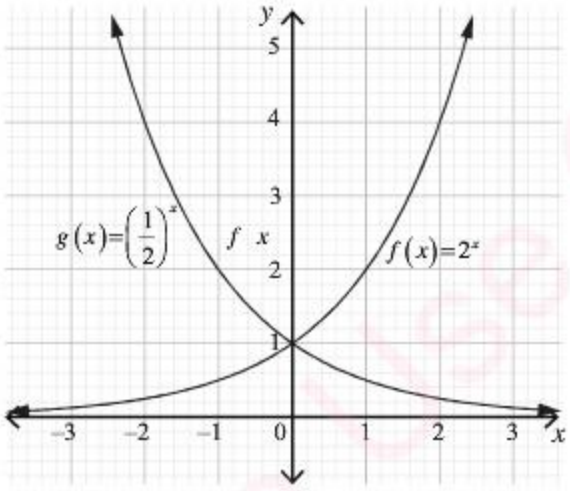
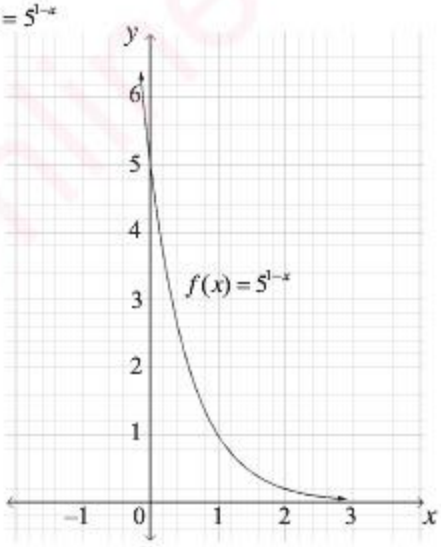
c) Domain of $f(x) = \{x : x \in \mathbb{R}\}$ Domain of $g(x) = \{x : x \in \mathbb{R}\}$

Range of $f(x) = \{y : y > 0\}$ Range of $g(x) = \{y : y > 0\}$

2. The graph of $g(x) = \left(\frac{1}{2}\right)^x$ and $f(x) = 2^x$



3. The graph of $f(x) = 5^{1-x}$



Domain of $f(x) = \{x : x \in \mathbb{R}\}$ Range of $f(x) = \{y : y > 0\}$

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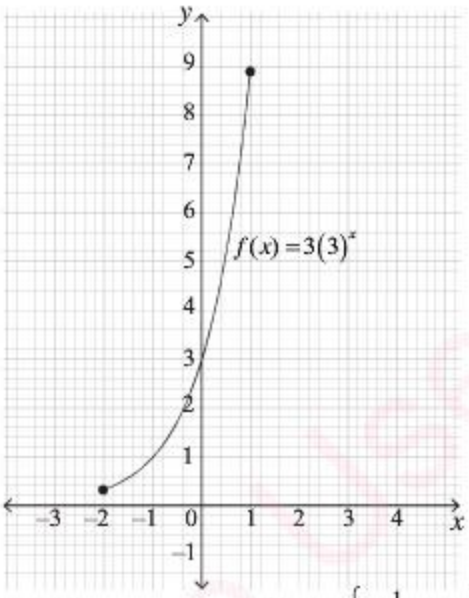
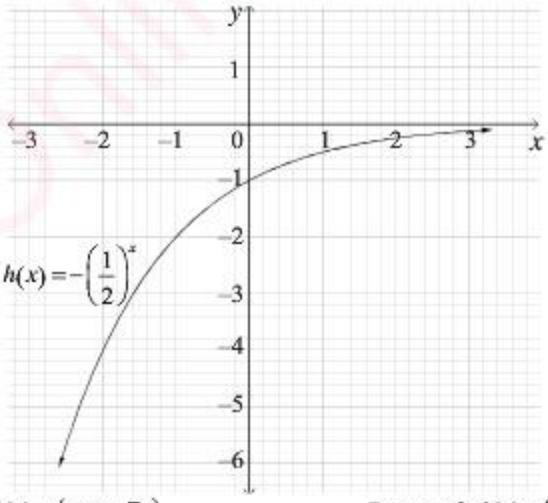
4. The graph of $f(x)=3(3)^x$



Domain $=\{x:-2\le x\le 1\}$ Range $=\left\{y:\frac{1}{3}\le y\le 9\right\}$

5. The graph of $h(x)=-\left(\frac{1}{2}\right)^x$



Domain of $f(x)=\{x:x\in\mathbb{R}\}$ Range of $f(x)=\{y:y<0\}$

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Exercise 9.3

1. a) $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ b) $3e^{3x-4}$ c) $\left[\frac{1}{2\sqrt{x}}+6x\right]e^{\sqrt{x}+3x^2}$
2. a) 2.04 b) 504.29 c) 1.12
3. a) a^2e^{ax+b} b) $6(2x^2+1)e^{1+x^2}$ c) $(\cos^2 x-\sin x)e^{\sin x}$
4. $-\frac{1}{6}e^{5t}$

Exercise 9.4

1. a) $\frac{1}{5}e^{5x}+c$ b) $-\frac{1}{9}e^{-9x+1}+c$
   c) $\frac{1}{3}e^{6+3x}+c$
2. a) 0.10 b) 1.72
   c) 4602.96
3. $\frac{1}{2}e^{x^2+4x+3}+c$ 4. 7.215

Exercise 9.5

1. a) $\log_a\left(x^p y^{\frac{1}{q}}\right)$ b) $\ln\left(\frac{\sqrt{x-2}}{x}\right)$ c) $\log\left[(x+1)^4(x-1)^8\right]$
2. a) 2 b) $x-1$ c) 0
3. a) –1.48 b) 0.16 5. $y=\frac{a^2}{b}$

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Exercise 9.6

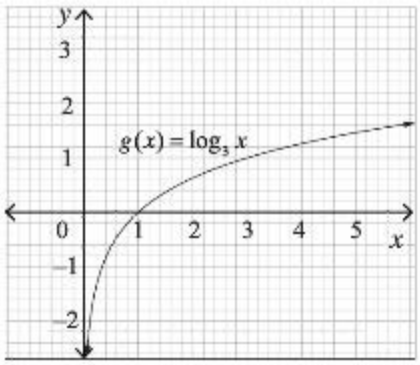
1. Domain = $\{x : x > -2\}$

   Range = $\{y : y \in \mathbb{R}\}$

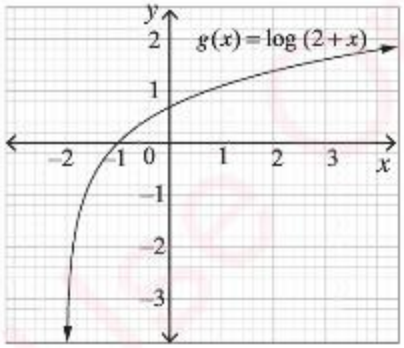
2. Domain = $\{x : x < 1\}$

   Range = $\{y : y \in \mathbb{R}\}$

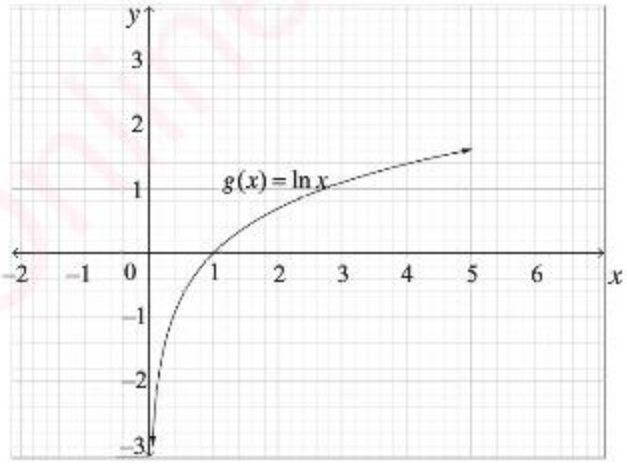
3. The graph of $g(x) = \log_3 x$



4. The graph of $g(x) = \log(2 + x)$



5. The graph of $g(x) = \ln x$



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Exercise 9.7

1. a) $\frac{9}{9x+7}$ b) $\frac{-1}{x+2}$ c) $\frac{1}{(x+1)(x+2)}$

2. a) $\frac{1}{(x+1)\ln 10}$ b) $\frac{4}{\ln 10}$ c) $\frac{2x}{(x^2+2)\ln 10}$

3. a) $3^x \ln 3$ b) $5^{2x}\ln 25$ c) $100^{-x}\ln\left(\frac{1}{100}\right)$ *or* $-100^{-x}\ln 100$

4. a) $2x\ln(2x)+x$ b) $x^2 2^{x+1}(3+x\ln 2)$ c) $\frac{3^x(x\log x\ln 3\ln 10-1)}{x\ln 10(\log x)^2}$

5. a) $-3^{-x}\ln 3;\ 3^{-x}(\ln 3)^2$ b) $-\frac{1}{(1-x)\ln 10};-\frac{1}{(1-x)^2\ln 10}$ c) $\frac{1}{x+2};-\frac{1}{(x+2)^2}$

Exercise 9.8

1. $\frac{1}{5}\ln|5x-1|+c$
2. $\frac{1}{10}\ln|10x-3|+c$
3. $\frac{1}{3}\ln|3x+7|+c$
4. $\frac{1}{5}\ln|5x+6|+c$
5. a) -0.35
   b) -0.02

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Exercise 9.9

1. Tsh 70,312.5
2. 44.11%
3. 3.1 years
4. Tsh 500,000
5. 11.20 years
7. a) Tsh 198,932.49
   b) Tsh 66136.10
6. Tsh 1,495,516.91

Exercise 9.10

1. $k = 0.0139$
2. a) $P(t) = 500,000e^{0.2t}$
   b) 1.1013×10^{10} people
3. 2014 lizards

Exercise 9.11

1. a) $N(t) = 800e^{-0.1386t}$
   b) 12.5 milligrams
   c) 48.2 days
2. 17,522.8 years
3. 160.11 grams
4. a) $N(t) = N_0 e^{-0.0555t}$
   b) 5.4%
   c) 0.7577 milligrams
5. a) $N(t) = N_0 e^{-0.00042t}$
   b) 0.042%
   c) 0.9916grams

Exercise 9.12

1. 297.11 K
2. 126.9 minutes
3. 332.07 K
4. 78.40 minutes
5. 0.0773 per second

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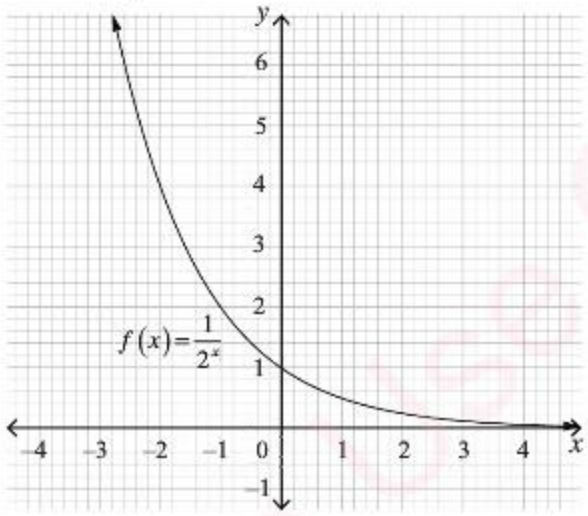
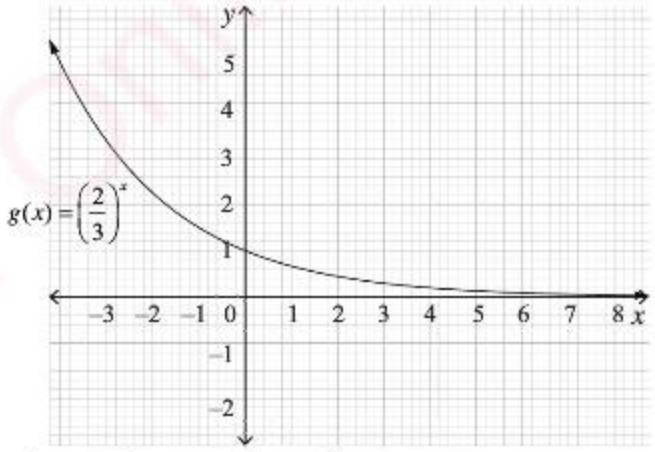
Revision Exercise 9

1. The graph of $f(x)=\frac{1}{2^x}$



Domain $=\{x: x\in\mathbb{R}\}$, Range $=\{y: y>0\}$

b) The graph of $g(x)=\left(\frac{2}{3}\right)^x$



Domain $=\{x: x\in\mathbb{R}\}$, Range $=\{y: y>0\}$

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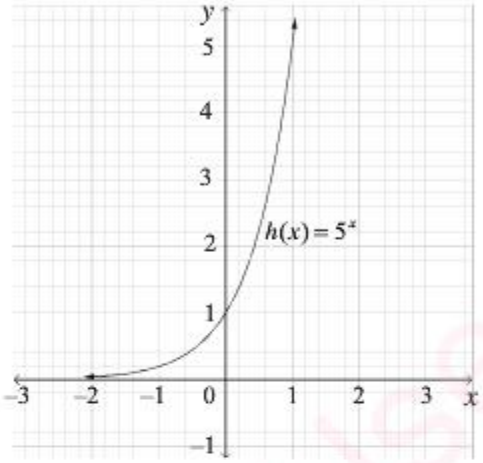
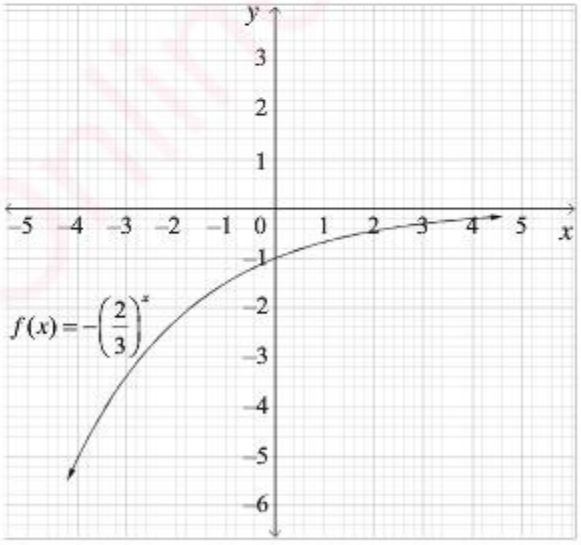
c) The graph of $h(x) = 5^x$



Domain = $\{x : x \in \mathbb{R}\}$, Range = $\{y : y > 0\}$

d) The graph of $f(x) = -\left(\frac{2}{3}\right)^x$



Domain = $\{x : x \in \mathbb{R}\}$, Range = $\{y : y < 0\}$

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2. a) $x=0$ b) $x=1$ c) $x=1$ or $x=4$

3. a) $2x(1+x)e^{2x}$ b) $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$ c) $6xe^{3x^2}$

d) $-4e^{-4x+2}$ e) $e^{3x}(3x+1)$

4. a) 0.4323 b) 0.4269 c) 0.8473

d) 0.5878

5. a) $\dfrac{6}{5}\ln|5x+1|+c$ b) $\ln|x+1|+c$ c) $\dfrac{3}{2}\ln|2x+3|+c$

6. Tsh 81,930.82 7. Tshs. 234,331.88 8. 602,249 bacteria

9. $P(t)=100e^{0.2506t}$ 10. a) 139,706.70 b) 139,996.80 11. a) 5,807,194 bacteria b) 15.36 hours

CHAPTER TEN

Exercise 10.1

1. a)

	Counter books	Text books	Calculators
Ally	8	6	2
Neema	7	9	1
Sauda	10	5	3

b) The order of the matrix is 3×3

2. a) The order of the matrix is 3×2 b) 6 elements

3. $x=2$, $y=-8$, $w=5$, and $z=8$

4. a) $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$ b) $\begin{bmatrix}14&0&0\\0&14&0\\0&0&14\end{bmatrix}$

5. a) Column matrix of order 3×1 b) Row matrix of order 1×3 c) 2×3 matrix

d) A square matrix of order 2×2 e) Scalar matrix of order 2×2

f) Square matrix of order 3×3

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Exercise 10.2

1. a) $\begin{pmatrix} 3 & 9 & 11 \\ 8 & 5 & 9 \end{pmatrix}$ b) $\begin{pmatrix} 3 & -5 & -1 \\ 4 & -5 & -7 \end{pmatrix}$

2. $\boldsymbol{P} = \begin{pmatrix} 7 & 12 \\ 7 & 12 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} 3 & -4 \\ 5 & -6 \end{pmatrix}$

4. $\boldsymbol{C} = \begin{pmatrix} -3 & -6 & -6 \\ 2 & 7 & -5 \\ -13 & -4 & -7 \end{pmatrix}$

5. $\begin{pmatrix} 19 & 11 & -6 \\ 12 & -4 & -9 \\ 0 & -6 & 10 \end{pmatrix}$

Exercise 10.3

1. $\begin{pmatrix} 29 & 20 \\ 19 & 30 \end{pmatrix}$

2. $\begin{pmatrix} -\frac{55}{2} & 1 \\ \frac{3}{2} & -26 \end{pmatrix}$

3. $\begin{pmatrix} -8 & 18 & 70 \\ -11 & 42 & 34 \\ 51 & 56 & 26 \end{pmatrix}$

4. $a = 4$, $b = 10$, $c = 7$, and $d = -12$

5. $t = -2$, $u = -6$

Exercise 10.4

1. a) $\begin{pmatrix} 14 & 50 & 83 \\ 23 & 85 & 138 \\ 31 & 95 & 145 \end{pmatrix}$ b) $\begin{pmatrix} 69 & 75 & 82 \\ 91 & 98 & 105 \\ 77 & 74 & 77 \end{pmatrix}$ c) $\boldsymbol{AB} \neq \boldsymbol{BA}$, multiplication of matrices not commutative.

3. a) $\boldsymbol{M} = \begin{pmatrix} 60 & 80 \\ 30 & 100 \end{pmatrix}$ b) $\boldsymbol{N} = \begin{pmatrix} 12000 \\ 10000 \end{pmatrix}$, $\boldsymbol{MN} = \begin{pmatrix} 1520000 \\ 1360000 \end{pmatrix}$

5. $\boldsymbol{AB} = \begin{pmatrix} 32 \\ 76 \end{pmatrix}$. That is, the number of instruments M produced by the factories is 32 and of the instruments N is 76

6. $\boldsymbol{LN} = \begin{pmatrix} 40 & 24 \\ 34 & 35 \end{pmatrix}$, $\boldsymbol{NL} = \begin{pmatrix} 21 & 24 & 9 \\ 13 & 18 & 13 \\ 32 & 46 & 36 \end{pmatrix}$

$\boldsymbol{LN} \neq \boldsymbol{NL}$. Hence, matrix multiplication is not commutative.

7. $a = -15$, $b = \frac{-51}{5}$

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Exercise 10.5

1. a) The order is 2×3 Transpose is $\begin{pmatrix}3&6\\2&0\\5&1\end{pmatrix}$

b) The order is 3×2 Transpose is $\begin{pmatrix}2&3&7\\0&1&2\end{pmatrix}$

c) The order is 2×2 Transpose is $\begin{pmatrix}7&8\\1&2\end{pmatrix}$

2. a) $\begin{pmatrix}2&1&3\\7&8&2\\-9&-2&0\end{pmatrix}$ b) $\begin{pmatrix}6&2\\5&8\\-5&-7\end{pmatrix}$ c) $\begin{pmatrix}1&2&4\\0&0&4\\5&5&7\end{pmatrix}$

3. $(\boldsymbol{AR})^T=\begin{pmatrix}15&5&25\\19&-3&13\\15&-3&10\end{pmatrix}$

5. $(\boldsymbol{GH})^T=\begin{pmatrix}66&-12\\188&-24\\104&-10\end{pmatrix}$

Exercise 10.6

1. $|\boldsymbol{A}|=-6$

2. $a=\pm4$

3. $k=\frac{2}{5}$

Exercise 10.7

1. The co-factor of a_{13} is 3

2. The minor of a_{31} is -38

3. $\begin{pmatrix}6&-12&-10\\1&-12&-5\\-8&-4&0\end{pmatrix}$

4. $\begin{pmatrix}6&12&-10\\-1&-12&5\\-8&4&0\end{pmatrix}$

5. The co-factor of -6 and -10 are -32 and -63, respectively.

Exercise 10.8

1. $a=3.28$ or $a=-0.61$ 2. a) -6 b) 25 c) -26 d) -23 4. $y=4$

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Exercise 10.9

1. $C^{-1} = \begin{pmatrix} -\frac{5}{3} & 2 \\ 1 & -1 \end{pmatrix}$

2. $A^{-1} = \begin{pmatrix} -1 & -\frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

3. $B^{-1} = \begin{pmatrix} -2 & 5 \\ 3 & -7 \end{pmatrix}$

4. $A^{-1} = \begin{pmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{pmatrix}$

5. $W^{-1} = \begin{pmatrix} \frac{4}{17} & \frac{3}{17} \\ -\frac{3}{17} & \frac{2}{17} \end{pmatrix}$

Exercise 10.10

1. a) $A^{-1} = \begin{pmatrix} \frac{-5}{79} & \frac{15}{79} & \frac{-2}{79} \\ \frac{24}{79} & \frac{7}{79} & \frac{-22}{79} \\ \frac{3}{79} & \frac{-9}{79} & \frac{17}{79} \end{pmatrix}$

b) $B^{-1} = \begin{pmatrix} \frac{1}{37} & \frac{21}{148} & \frac{3}{74} \\ \frac{12}{37} & \frac{-7}{148} & \frac{1}{74} \\ \frac{-3}{37} & \frac{11}{148} & \frac{9}{74} \end{pmatrix}_S$

c) $E^{-1} = \begin{pmatrix} 0 & \frac{-3}{14} & \frac{1}{7} \\ \frac{-1}{3} & \frac{-1}{42} & \frac{5}{21} \\ \frac{1}{3} & \frac{13}{42} & \frac{-2}{21} \end{pmatrix}$

d) $F^{-1} = \begin{pmatrix} \frac{-3}{13} & \frac{4}{13} & \frac{-3}{13} \\ \frac{-28}{39} & \frac{7}{39} & \frac{-2}{39} \\ \frac{20}{39} & \frac{-5}{39} & \frac{7}{39} \end{pmatrix}$

3. $D^{-1} = \begin{pmatrix} 1 & -\sin^2 x \cos x & -\cos^2 x \\ 0 & \cos x & -1 \\ 0 & \sin^2 x & \cos x \end{pmatrix}$

Exercise 10.11

1. $(x, y) = (5, 1)$ 2. $(x, y) = (1, 2)$ 3. $(x, y) = (6, 2)$

4. Ages of the father and son are 38 and 14 years respectively

5. 35 pieces of cakes and 52 bottles

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Exercise 10.12

1. $x=1, y=-1, z=-1$
2. $x=-\frac{2}{3},\ y=4\frac{1}{3},\ z=3\frac{2}{3}$
3. a) $A^{-1}=\begin{pmatrix} -\frac{1}{8} & \frac{17}{160} & \frac{11}{160} \\ -\frac{1}{4} & \frac{1}{80} & \frac{3}{80} \\ -\frac{1}{8} & -\frac{11}{160} & \frac{7}{160} \end{pmatrix}$

   b) $x=1, y=1, z=1$
4. 2 Units of Magnesium
   3 Units of Phosphorus
   4 Units of Potassium
   $x=2, y=3, z=1$

Exercise 10.13

1. $x=3, y=0$
2. $x=3, y=3$
3. $x=1, y=3$
4. Cost of one tomato is Tsh 100
   Cost of one onion is Tsh 200
5. 3 kg of rice, 3 kg of beans

Exercise 10.14

1. a) $x=3, y=1, z=2$

   b) $\ddot{u}=2\frac{2}{3},\ =-4,\ =1\frac{1}{3}$

   c) $\ddot{u}=1,\ =-1,\ =-1$

   d) $x=-\frac{2}{3}, y=4\frac{1}{3}, z=3\frac{2}{3}$
2. 5 units of P_1, 8 units of P_2, 8 units of P_3

Revision Exercise 10

1. i) $\begin{pmatrix} 4 & 4 & 6 \\ 0 & 2 & 2 \\ 8 & 2 & -2 \end{pmatrix}$ ii) $\begin{pmatrix} 2 & 0 & 0 \\ -4 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix}$

2. $\begin{pmatrix} -8 & -4 & 12 \\ 12 & 4 & 0 \\ 12 & 8 & -8 \end{pmatrix}$

3. a) $\begin{pmatrix} 22 & 13 & -33 \\ -2 & -1 & 3 \\ 3 & 1 & 6 \end{pmatrix}$ b) $\begin{pmatrix} 72 & 14 & -21 \\ -8 & -1 & 3 \\ -19 & -1 & 10 \end{pmatrix}$ c) $\begin{pmatrix} 24 & 8 & -3 \\ -23 & -3 & 9 \\ -31 & -9 & 6 \end{pmatrix}$

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4. $\begin{pmatrix} 17 & 2 & -6 \\ -4 & 4 & 6 \\ -2 & 6 & 9 \end{pmatrix}$

5. a) 90 b) –9 6. k = – 2

8. $\frac{1}{5}\begin{pmatrix} -1 & 9 & -3 \\ -2 & 3 & -1 \\ 2 & -13 & 6 \end{pmatrix}$ 9. $x=1, y=2, z=3$ 10. $x=1.45,\ y=3.7,\ z=5.6$

11. a) $x=1.5,\ y=2.5,\ z=3.5$

b) $x=0.8,\ y=1.8,\ z=1.6$

12. a) $\boldsymbol{PS} = \begin{pmatrix} 21 \\ 22 \\ 20 \end{pmatrix}$

Interpretation:
Team A has 21 points
Team B has 22 points
Team C has 20 points

b) Team B leads the league

CHAPTER ELEVEN

Exercise 11.1

1. Maximize : $f(x, y) = 13,000x + 23,000y$
subject to: $5x + 3y \le 80$
$2x + 2y \le 60$
$10x + 7y \le 190$
$x \ge 0,\ y \ge 0.$

2. Maximize : $f(x, y) = 600,000x + 500,000y$
subject to: $x + y \le 168$
$60,000x + 80,000y \le 2,000,000$
$x \ge 0,\ y \ge 0.$

3. Minimize : $f(x, y) = 6,000x + 10,000y$
subject to: $x + 2y \ge 10$
$2x + 2y \ge 12$
$3x + y \ge 8$
$x \ge 0,\ y \ge 0.$

4. Minimize : $f(x, y) = 30,000x + 35,000y$
subject to: $20x + 10y \ge 460$
$30x + 30y \ge 960$
$5x + 10y \ge 220$
$x \ge 0,\ y \ge 0.$

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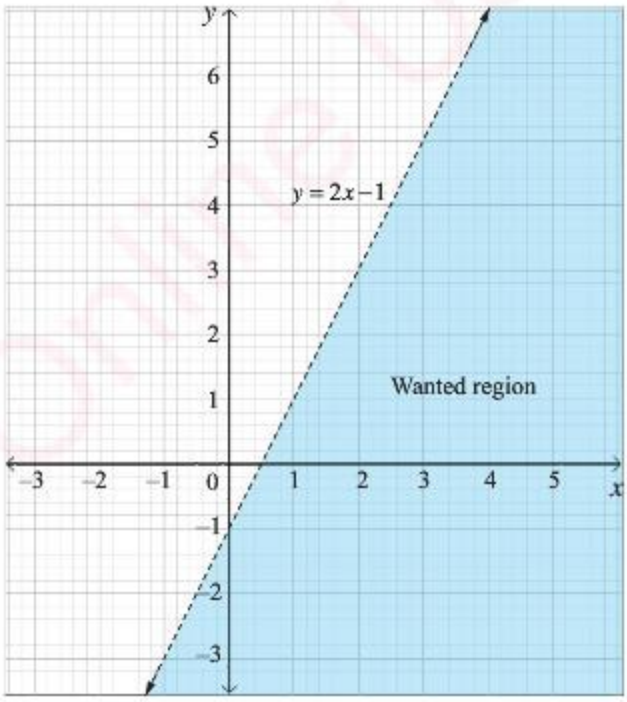
5. Maximize: $f(x, y) = x + y$
   subject to: $4x + y \leq 32$
   $x + 6y \leq 54$
   $x \geq 0,\ y \geq 0.$

6. Minimize: $f(x, y) = x + y$
   subject to: $6,000x + 5,000y \geq 30,000$
   $x \geq 1$
   $x \geq 0,\ y \geq 0.$

7. Maximize: $f(x, y) = 10x + 25y$
   subject to: $3 \leq x \leq 12$
   $4 \leq y \leq 15$
   $x + y \leq 20$
   $x \geq 0,\ y \geq 0.$

Exercise 11.2

1. a)



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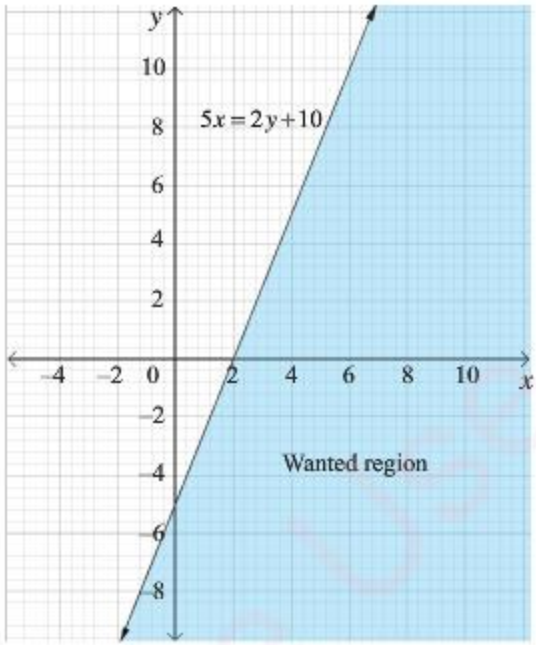
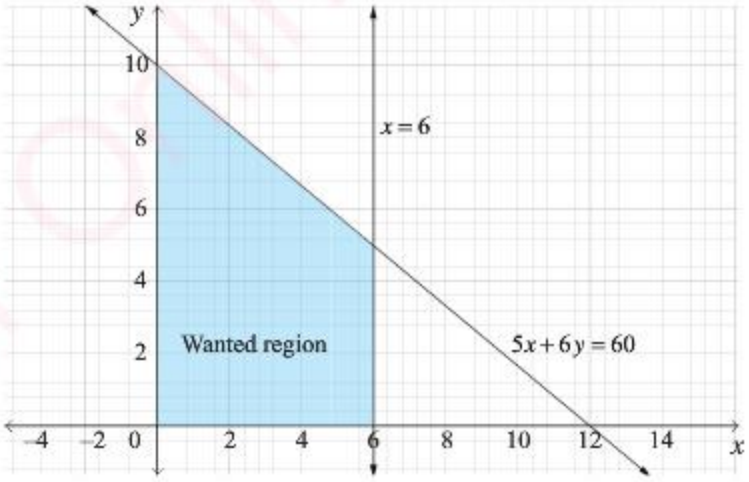
b)



2.



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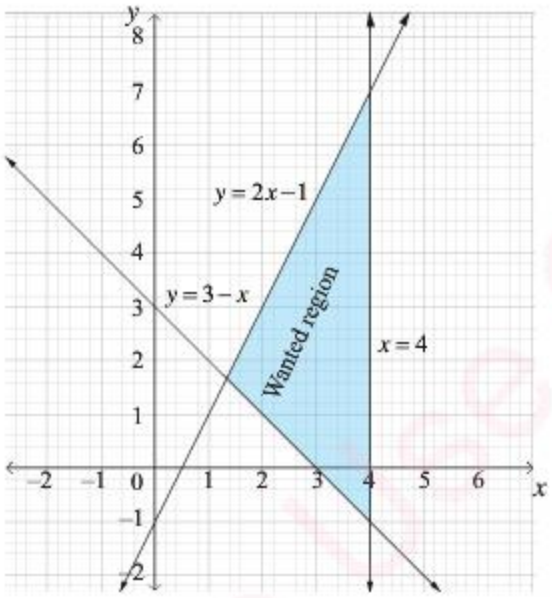
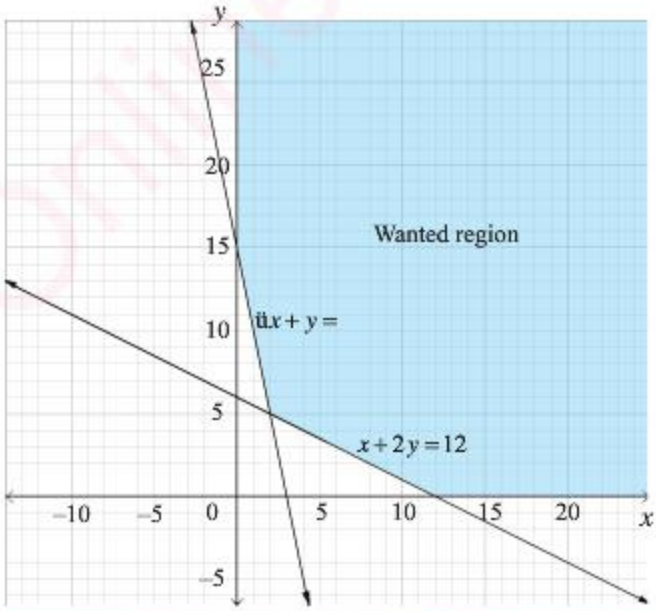
3. a)



b)



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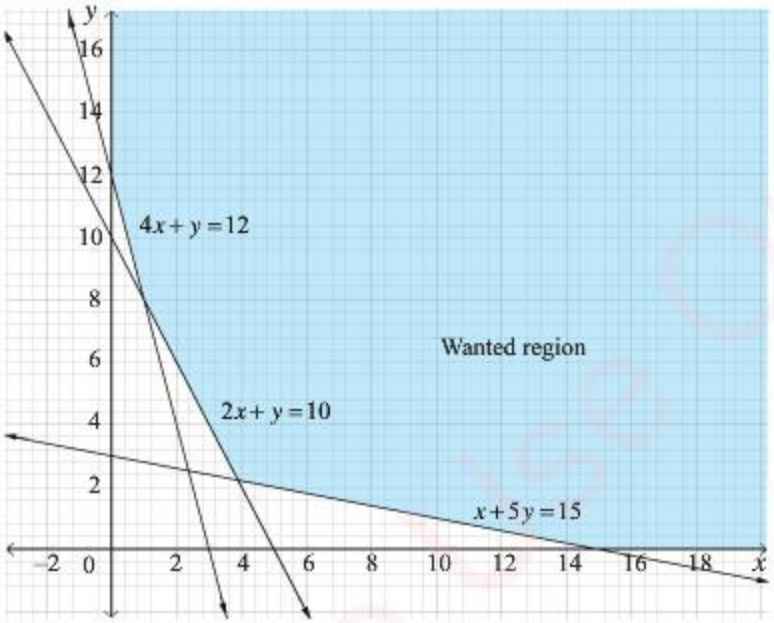
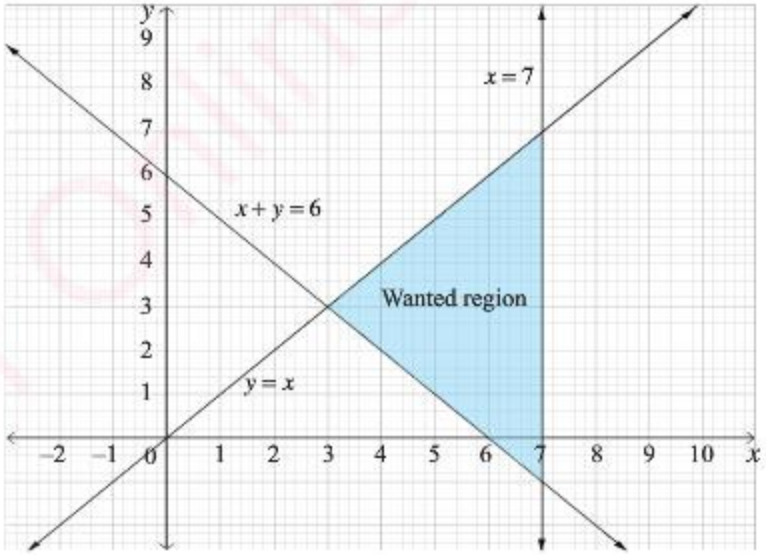
c)



4.



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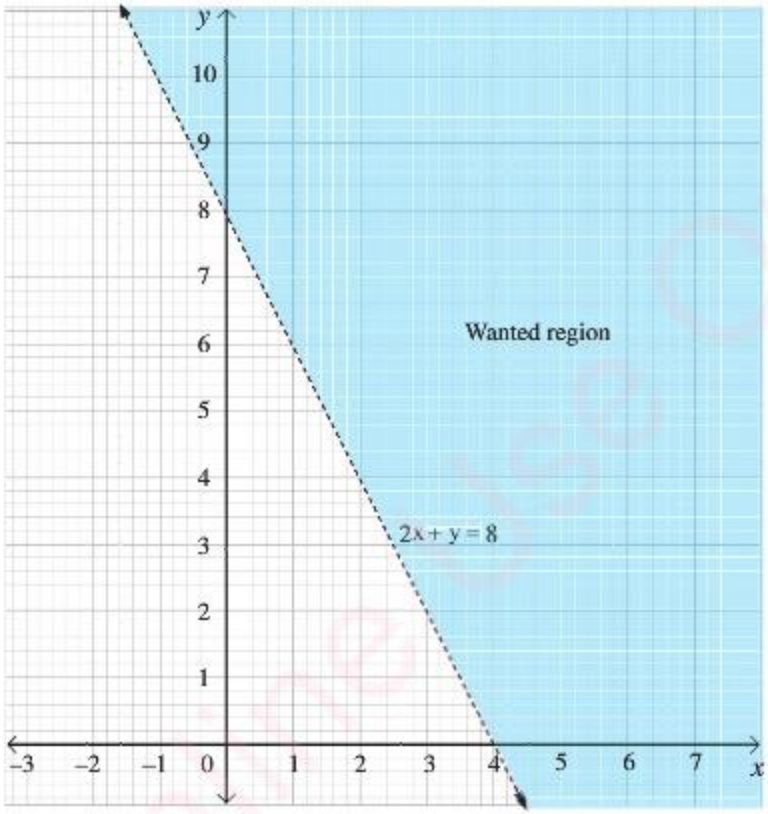
5.



Exercise 11.3

1. The maximum value of the objective function is 100 and it occurs when $x = 6$ and $y = 2$.
2. The minimum value is 40 and it occurs when $x = 10$ and $y = 0$.
3. The minimum value is 220 and it occurs when $x = 4$ and $y = 2$.
4. 5 hectares.
5. The lowest value is 59%. The corresponding values of x and y are $x = 24\%$ and $y = 35\%$
6. a) For maximum profit she should allocate 8 acres for maize and 5 acres for millet.

   b) The maximum profit is Tsh 219,000.

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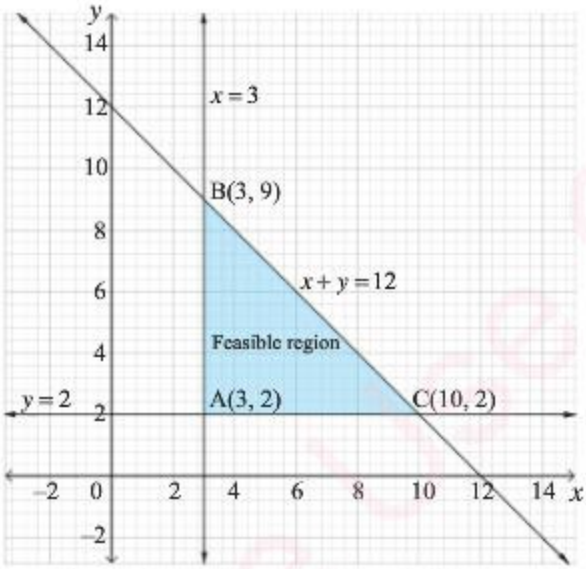
Revision exercise 11

1. Corner points: $(0,0)$, $(0,4)$, $(3,1)$ and $(3,0)$

3.



Hence, the maximum value is 54 and minimum is 19.

4. To maximize profit, 100 scientific calculators and 170 graphing calculators should be made.

5. To minimize cost, 3 bags of food I and 12 bags of food II should be ordered.

6. To minimize the total cost, he should buy 3 kg of food A and 2 kg of food B.

7. To maximize storage volume, she should buy 8 cabinets C_1 and 3 cabinets C_2.

8. To maximize gross income, the manufacturer should make 6 products A and 6 product B.

9. She should produce 6 machine A and 8 machine B.

The maximum profit is Tsh 374,000

10. To maximize profit, 8 tables and 6 chairs should be made.

11. To maximize loss of calories, they should spend 60 minutes on aerobics and 30 minutes on flexibility.

The maximum loss of calories is 570 calories.

12. To maximize profit, 9 acres of coffee and 4 acres of maize have to be planted.

a) The farmer should buy 40 bags of brand I and 20 bags of brand II to minimize the total cost.

b) The minimum total cost is Tsh 220,000.

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Glossary

Acceleration	A change of velocity with time
Combination	An arrangement of given number of objects into groups where the order of arrangement does not matter
Critical points	These are the turning points on the curve where the gradient changes sign from positive to negative (and vice versa) or the sign of the gradient remains constant. Example of critical points are the maximum point, minimum point and the point of inflexion.
Displacement	A distance travelled by a body in a specified direction
Joint variation	A type of variation in which one quantity depends directly on two or more other quantities
Permutation	An arrangement or listing of given number of objects in which the order of arrangement or listing matters
Preceding keys	These are keys of scientific calculators which need to be pressed first in order to access other scientific calculator functions. "SHIFT" and "ALPHA" keys are examples of preceding keys
Scale	A ratio between the distance on the map and the respective actual ground distance
Spreadsheet	A computer software designed specifically for performing calculations and drawing graphs
Velocity	A change of displacement with time

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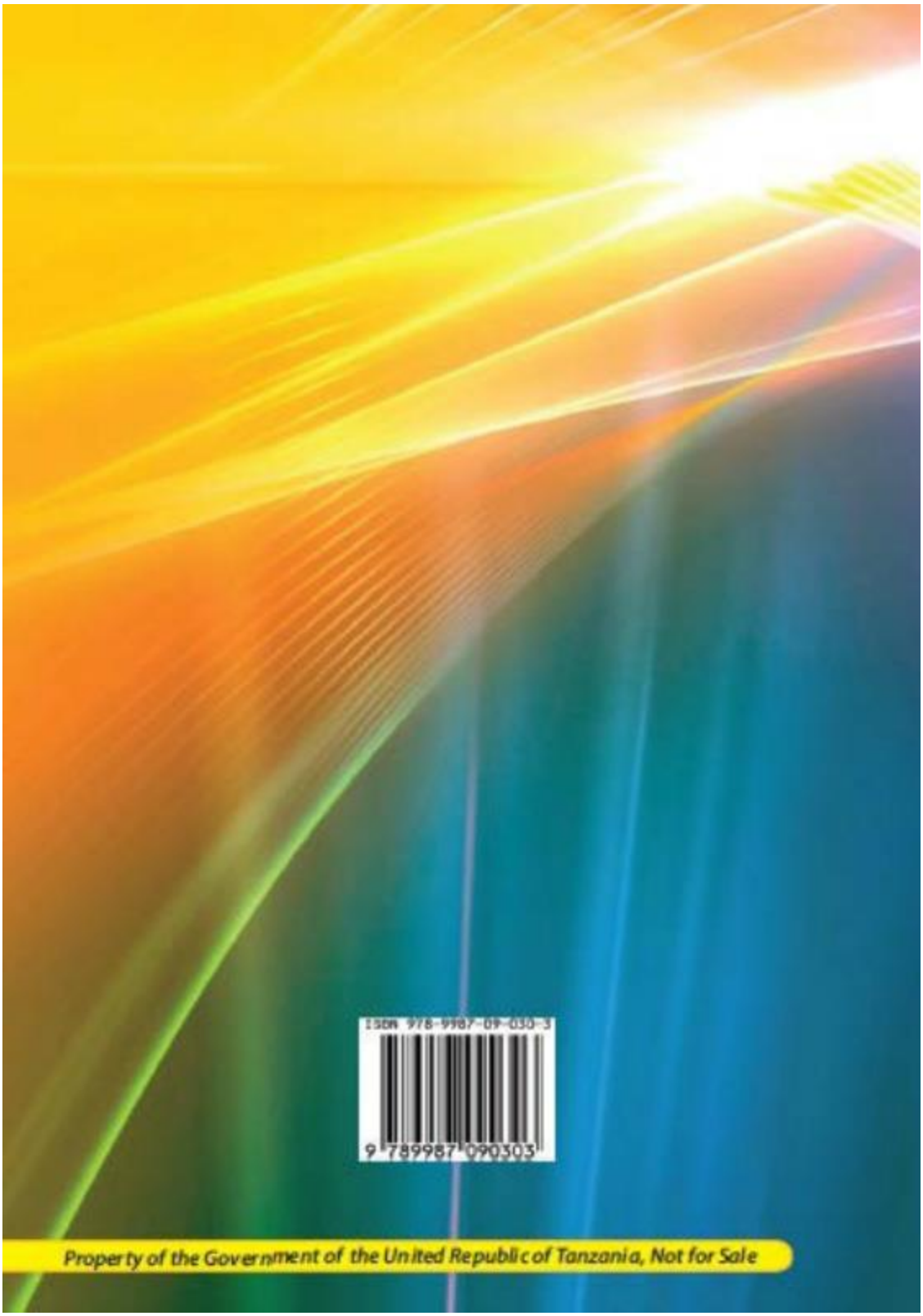
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